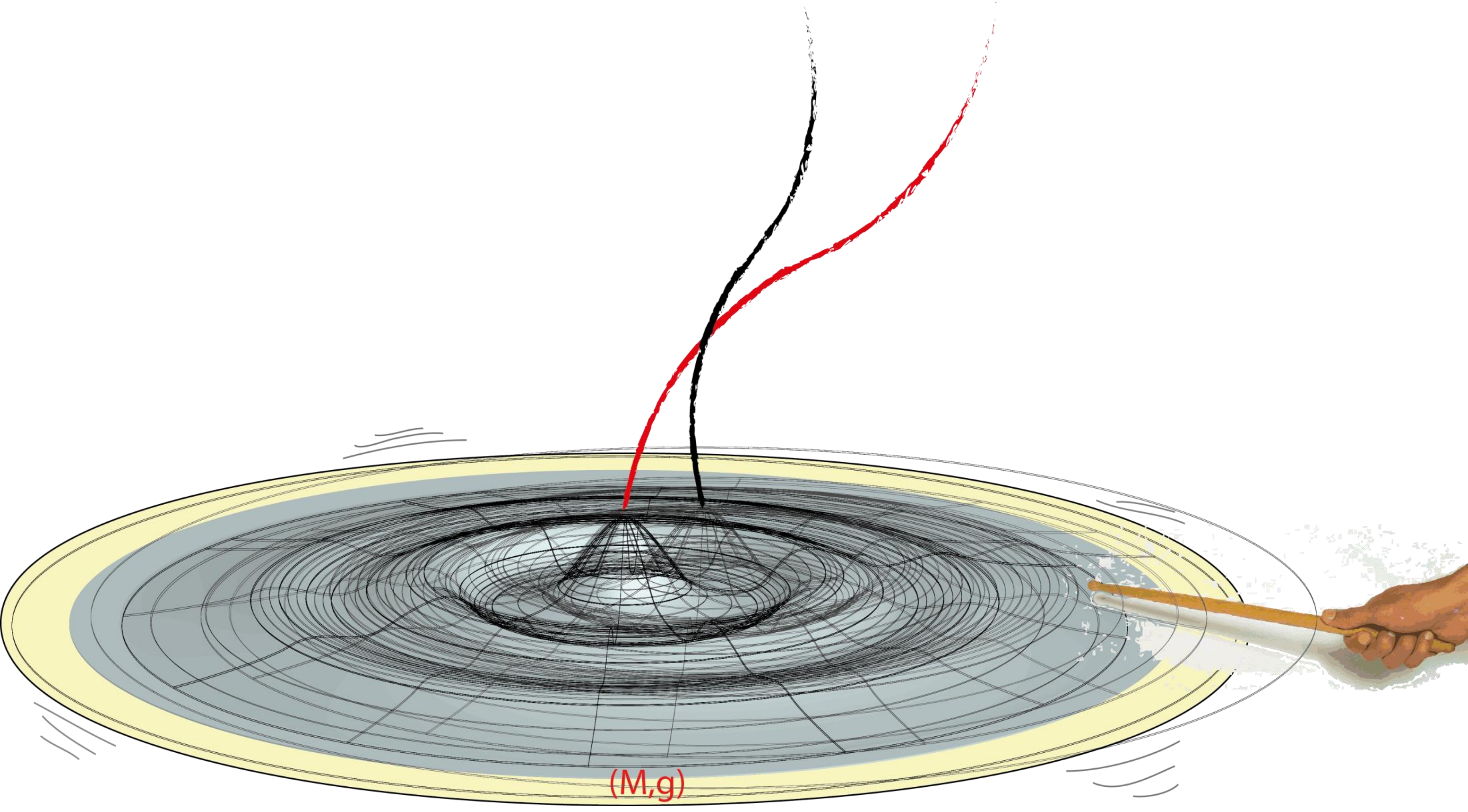
GRAVITATIONAL WAVES



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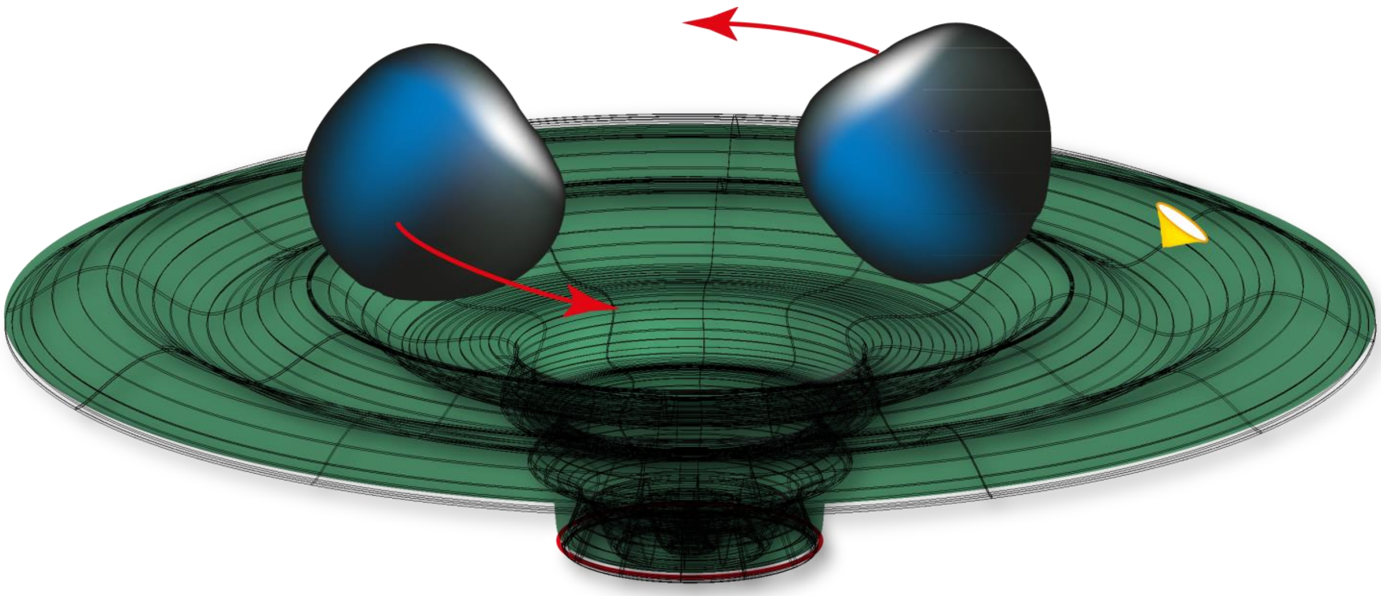
- The general theory of relativity is 100 years old
- It attributes gravity to the curvature of spacetime and it has been extremely successful.
- It has been tested with high precision in the solar system(*e.g.* Cassini, Bertotti and coll.) and in binary pulsars, (R. Hulse and J. Taylor, The Hulse–Taylor binary pulsar PSR1913+16).
- It explains the expansion of the universe, predict black holes and gravitational waves.
- We have seen them: A gravitational wave signal from two merging black holes



- The general theory of relativity is 100 years old (il 25 novembre 2015 Einstein illustró ai membri dell'Accademia Prussiana delle Scienze la sua teoria della Relativitá Generale e ne spiegava l'applicazione al fenomeno della precessione del perielio di Mercurio. Quattro anni piú tardi una delle predizioni della sua teoria- la deviazione dei raggi luminosi da parte del campo gravitazionale solare fu confermata dal team di scienziati (guidato da Sir Arthur Eddington) durante una eclisse solare totale (osservata nell'isola Principe). Einstein divenne, nello spazio di una giornata, una celebritá internazionale, una icona della fisica. Dalla metá degli anni 20 alla metá degli anni 50 la teoria giacque dormiente.

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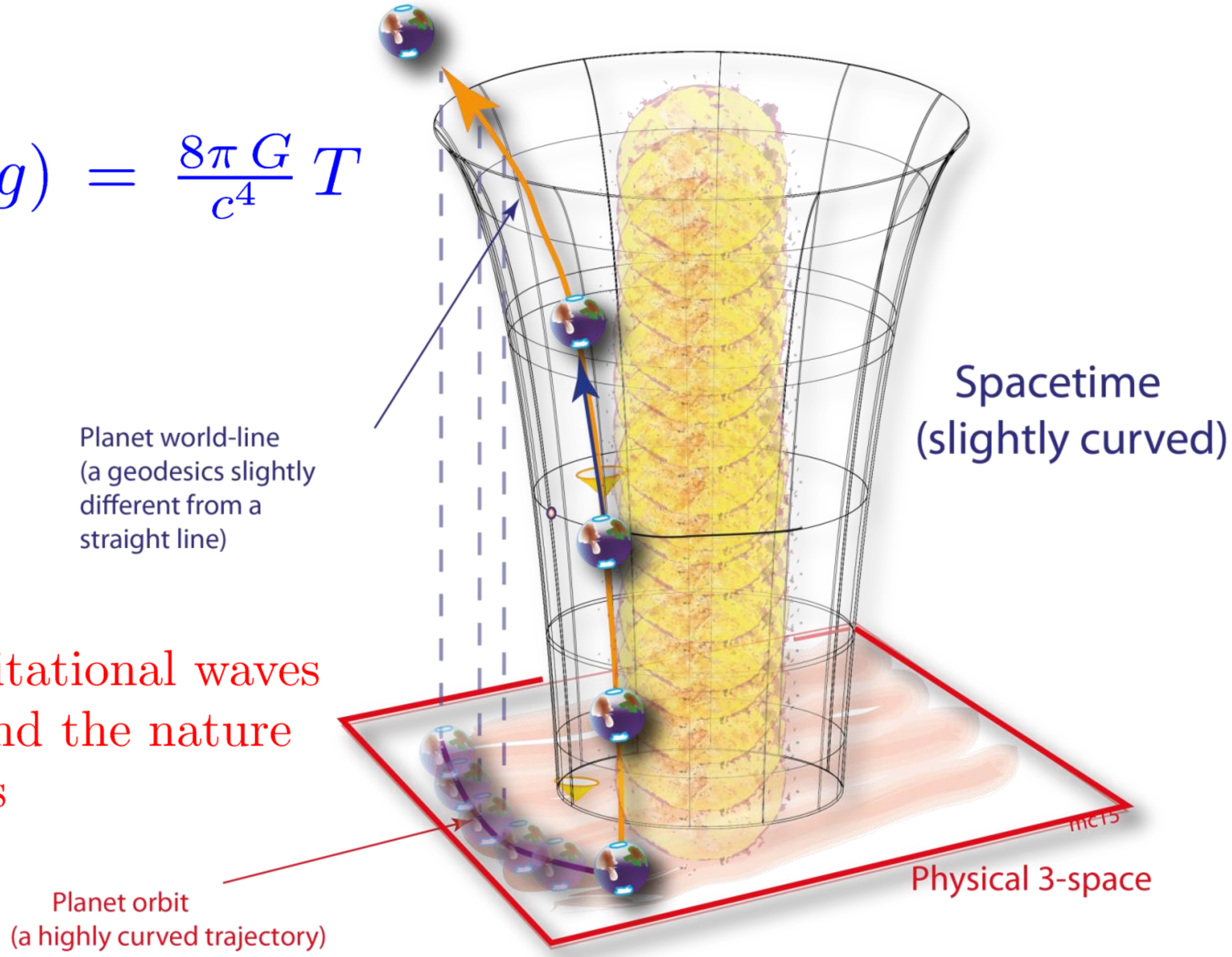
- The most fundamental aspect of GR is the blending of Space, Time and Gravitation in the geometry of a 4–dimensional curved Spacetime.

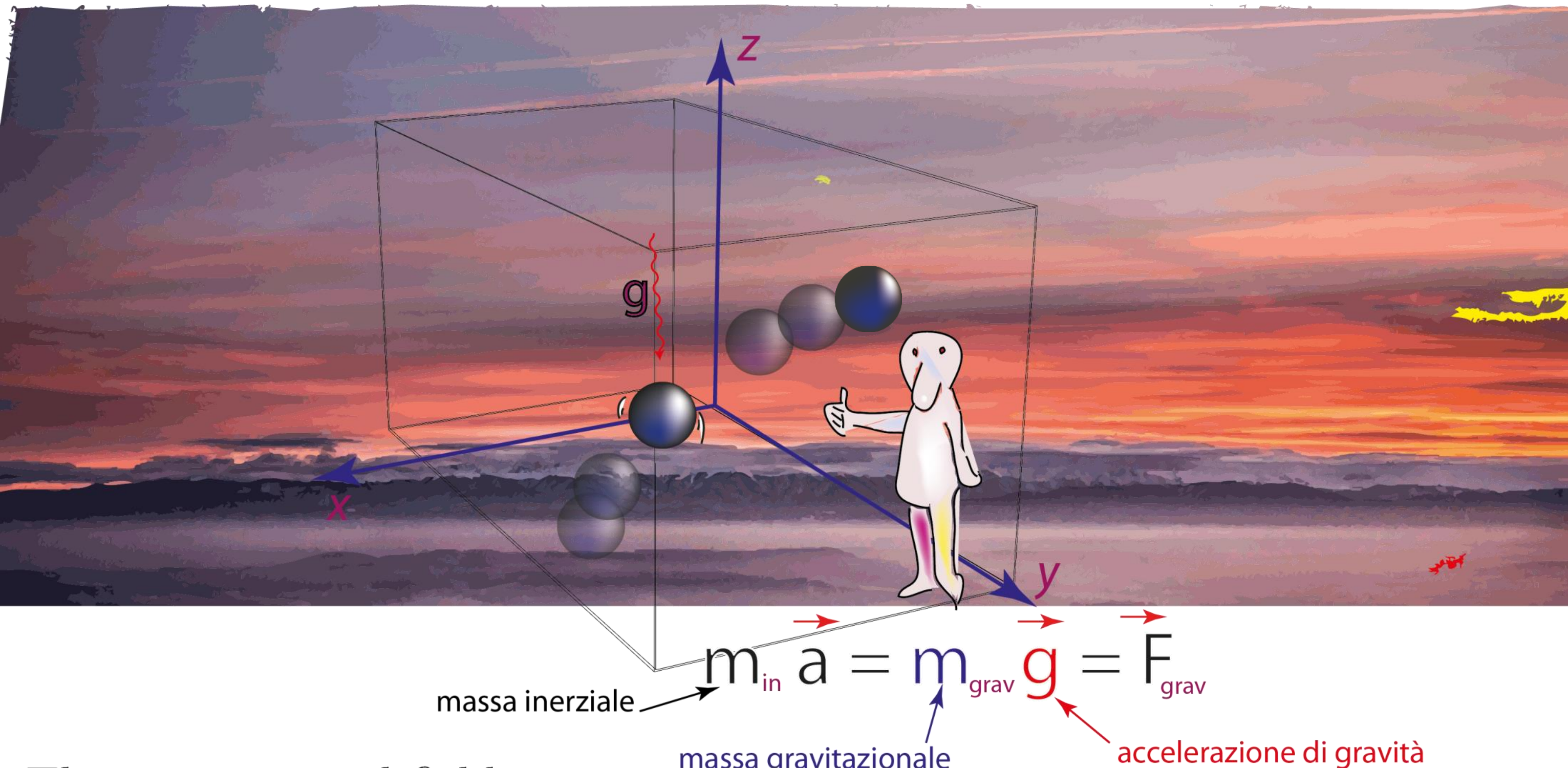
- Gravity becomes geometry via Einstein's equations that are imposed on the geometric structure of spacetime.

$$Ric(g) - \frac{1}{2}\, g\, R(g) = \frac{8\pi\, G}{c^4}\, T$$



- To understand Gravitational waves we need to understand the nature of Einstein equations

The gravitational field has a peculiar feature consequence of Newton's second law



The gravitational field is locally eliminable.

With respect to a free–fall elevator:

$$m_{in}\,(\vec{a} + \vec{g}) = m_{grav}\,\vec{g} + \vec{F}_{el}$$

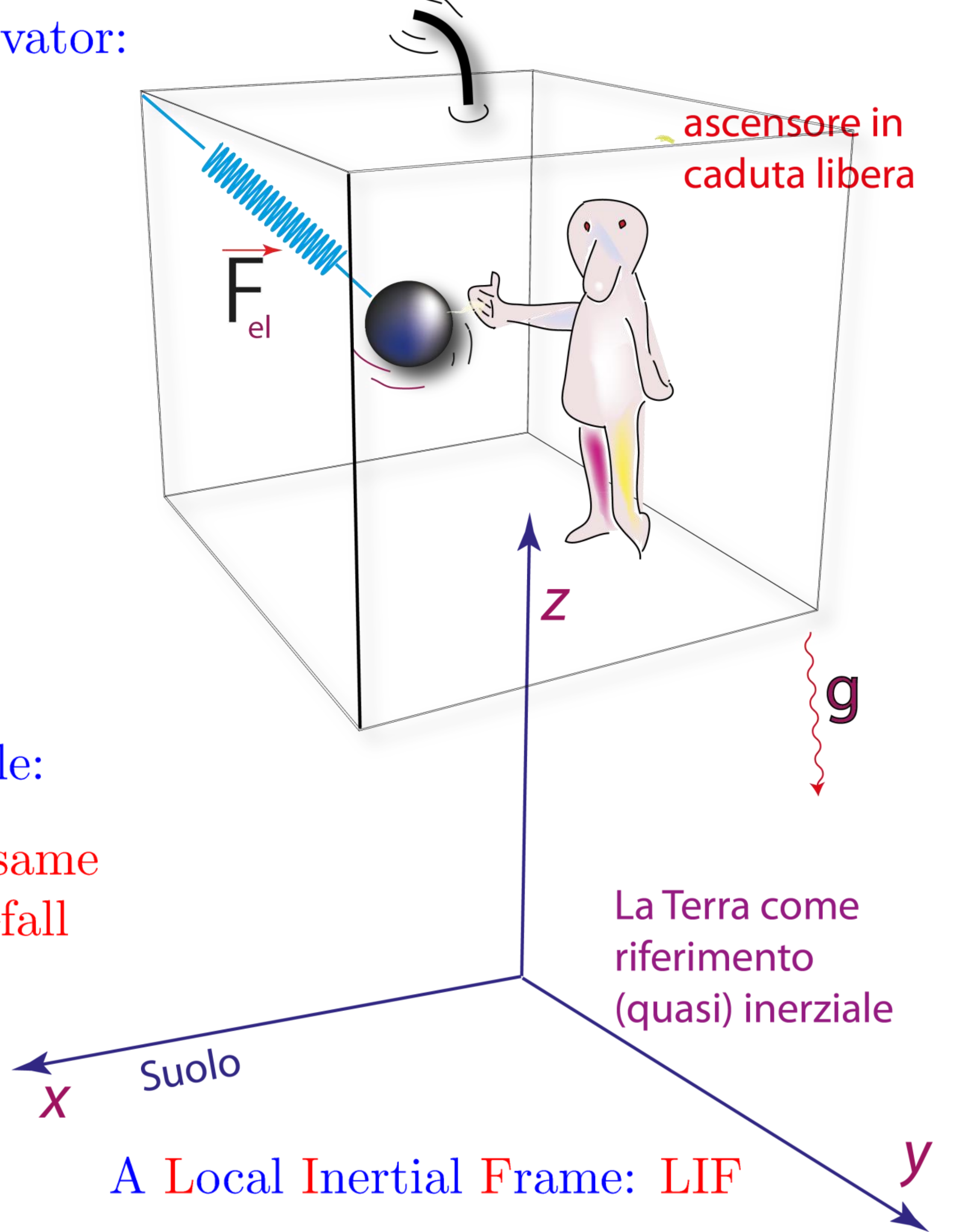
If $m_{in} = m_{grav}$, (Ëtvos)

$$m_{in}\,\vec{a} = \vec{F}_{el}$$

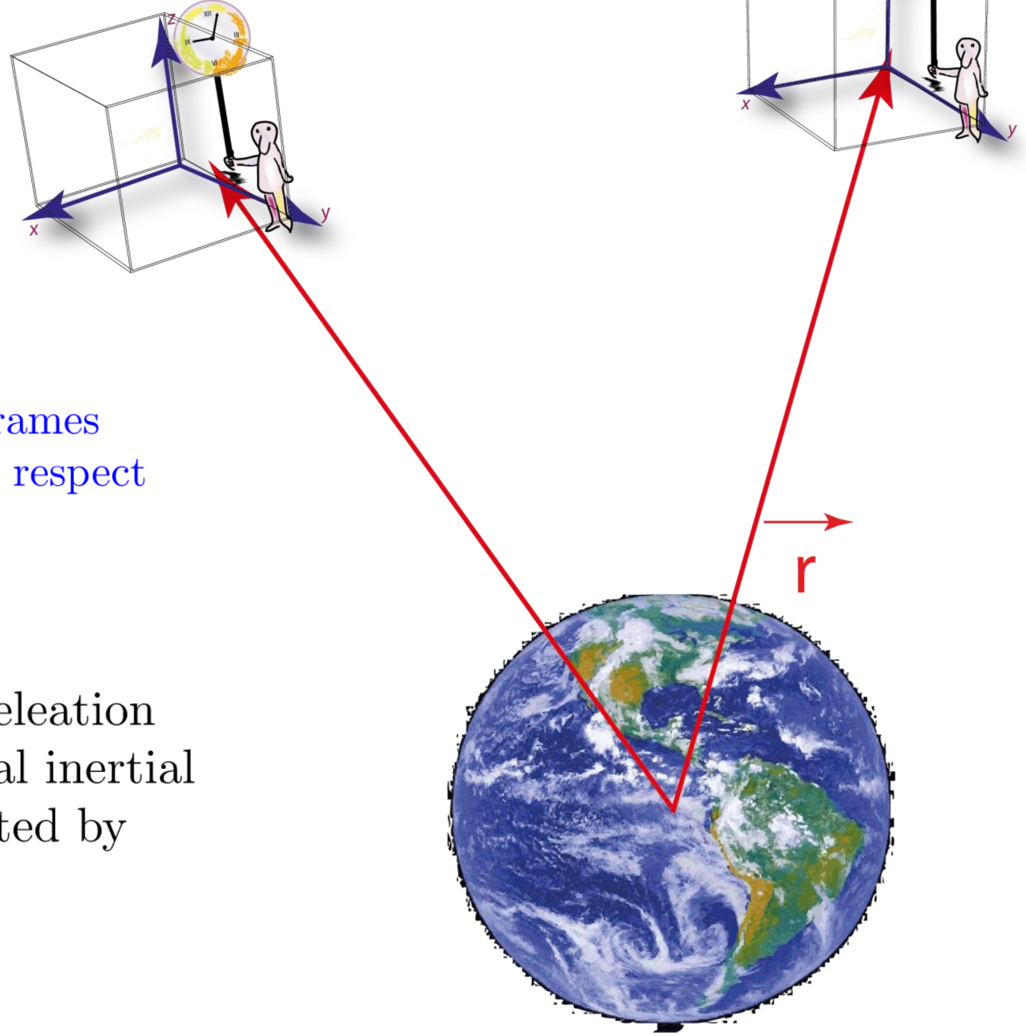
In a free–fall elevator the equations of motion take the same form they have in an inertial frame in absence of gravity

Einstein's equivalence principle:

The laws of physics take the same form in all non–rotating free–fall local frames : The LIFs



A Local Inertial Frame: LIF



In each local inertial frame

Special Relativity
Classical Physics in absence of gravity

The local inertial frames can accelerate with respect to each other.

The relative acceleration between two local inertial frames is generated by $\frac{\partial}{\partial x^k} G(\vec{r})$,

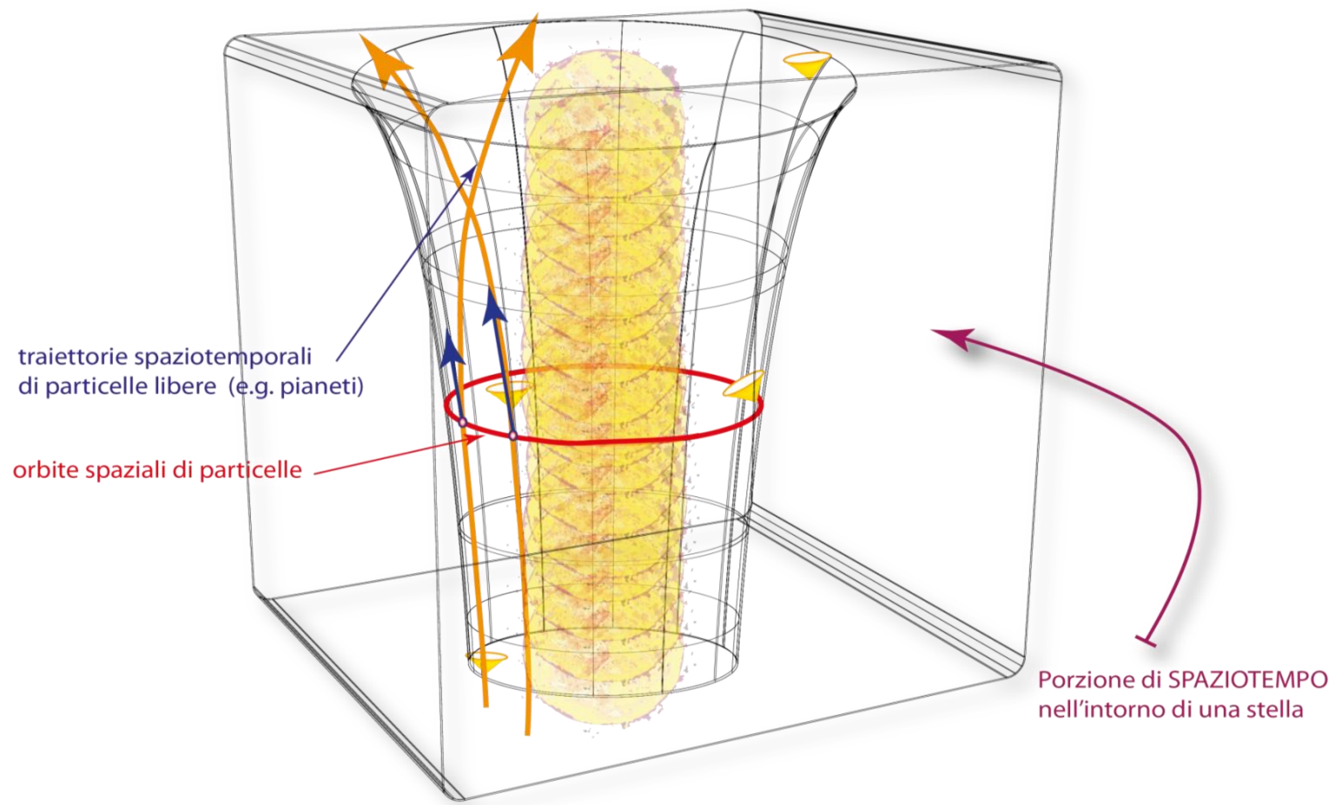
(Tidal Forces)

Tidal Forces: signature of the existence of non constant gravitational fields.

$$\vec{G}_{Newt}(r) = - \frac{G\, M_T}{r^3} \vec{r}$$

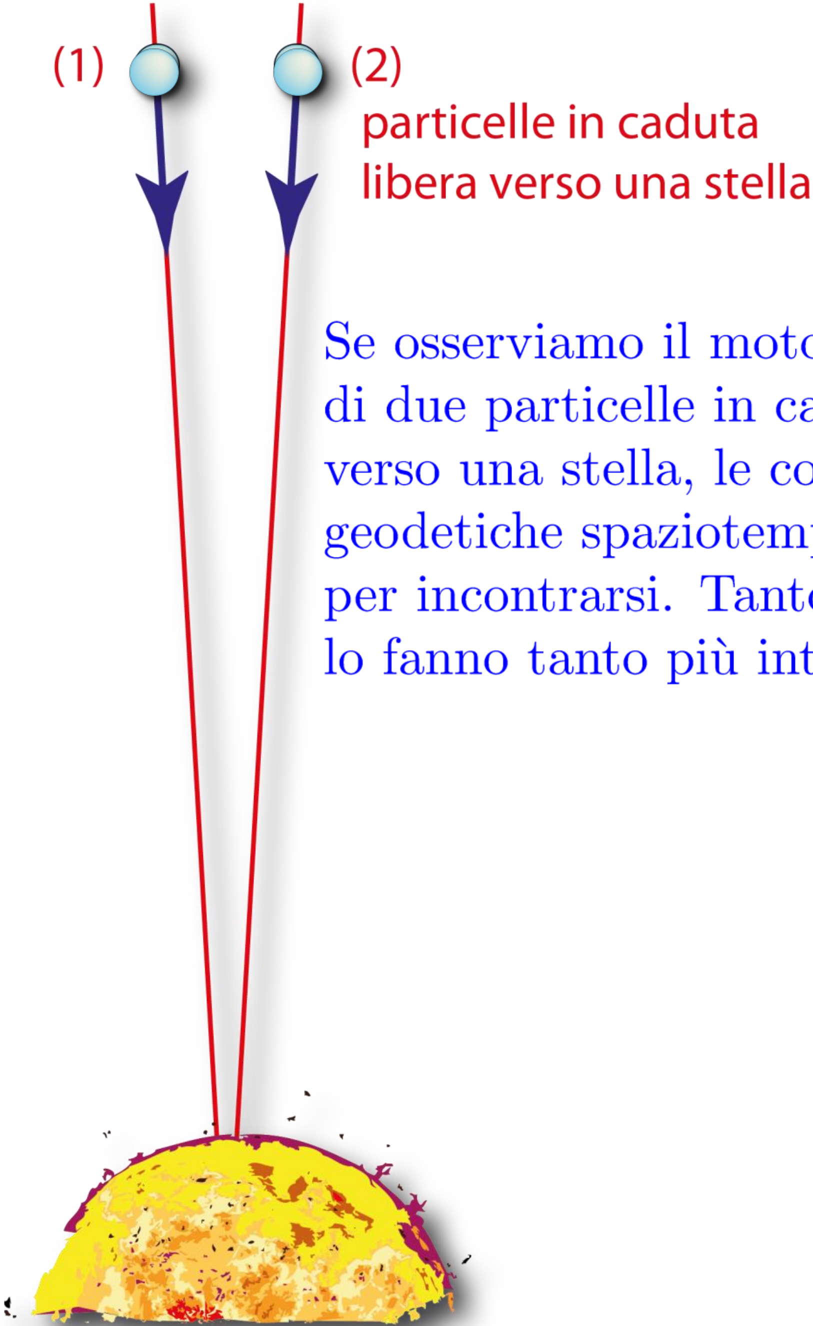
in modulo $|G|_{Newt} = \frac{G\, M_T}{r^2} \Big|_T \simeq g$

In Special Relativity the trajectories of photons and free particles are straight lines. However the Equivalence Principle implies that light, (and any physical motion), in presence of a gravitational field, follows a curved path.

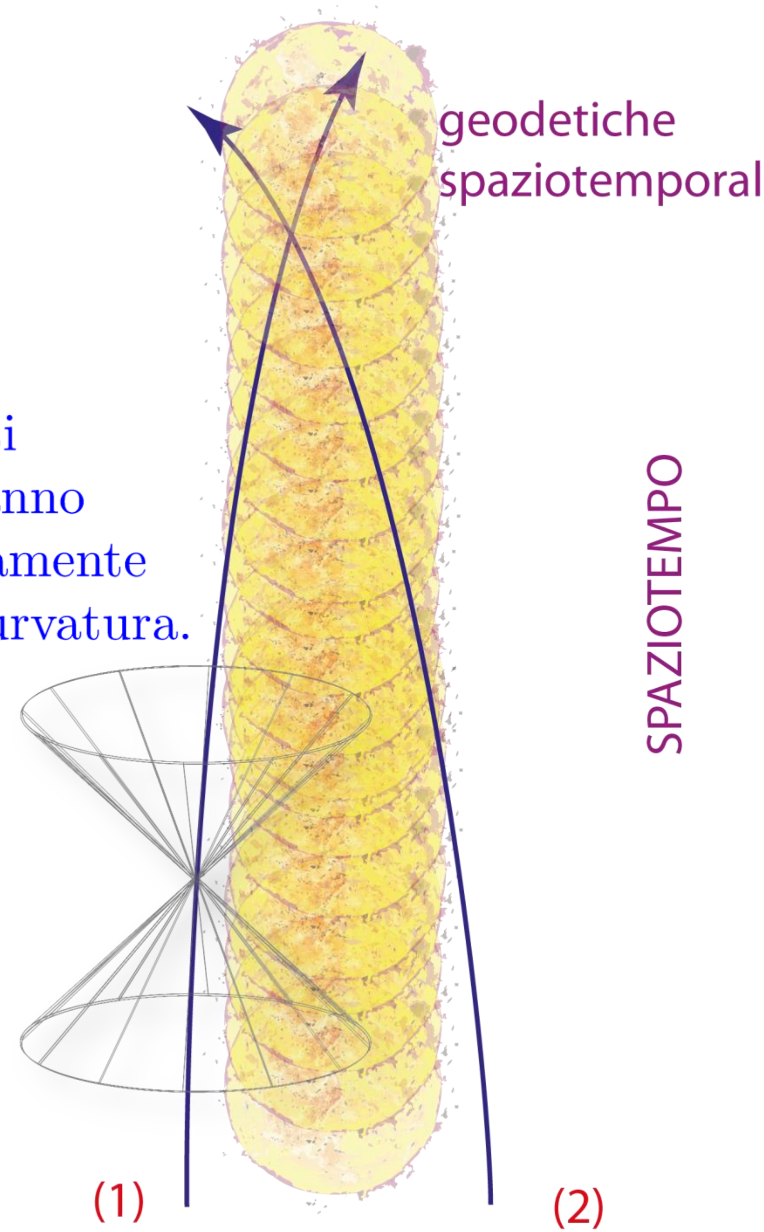


This is a strong indication that the bending of light rays in presence of a gravitational field is not a peculiar property of light, but rather of the interaction between the geometry of spacetime and its mass–energy content: Spacetime is Curved.

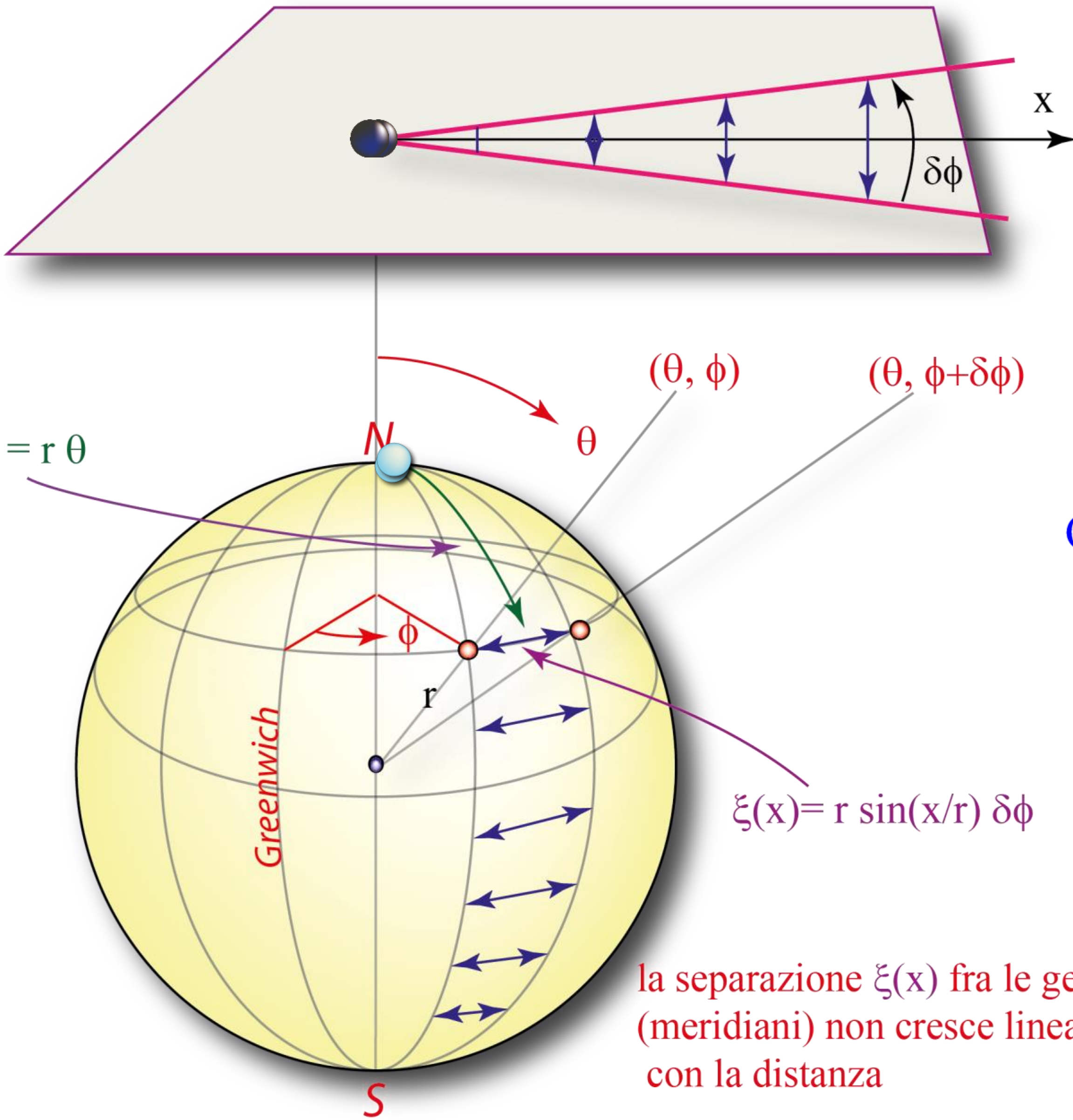
How can we describe the curvature of spacetime?



Se osserviamo il moto relativo di due particelle in caduta libera verso una stella, le corrispondenti geodetiche spaziotemporali finiranno per incontrarsi. Tanto più rapidamente lo fanno tanto più intensa è la curvatura.



la separazione s(x) fra le geodetiche (rette)
cresce linearmente con la distanza s(x)= x δϕ



la separazione ξ(x) fra le geodetiche
(meridiani) non cresce linearmente
con la distanza

Recall that in space geometry we have:

Relative acceleration between two geodesics (geodesic deviation):

On the plane: $\frac{d\,s(x)}{dx} = \delta\phi$, $\frac{d^2\,s(x)}{dx^2} = 0.$

On the 2–sphere of radius r:

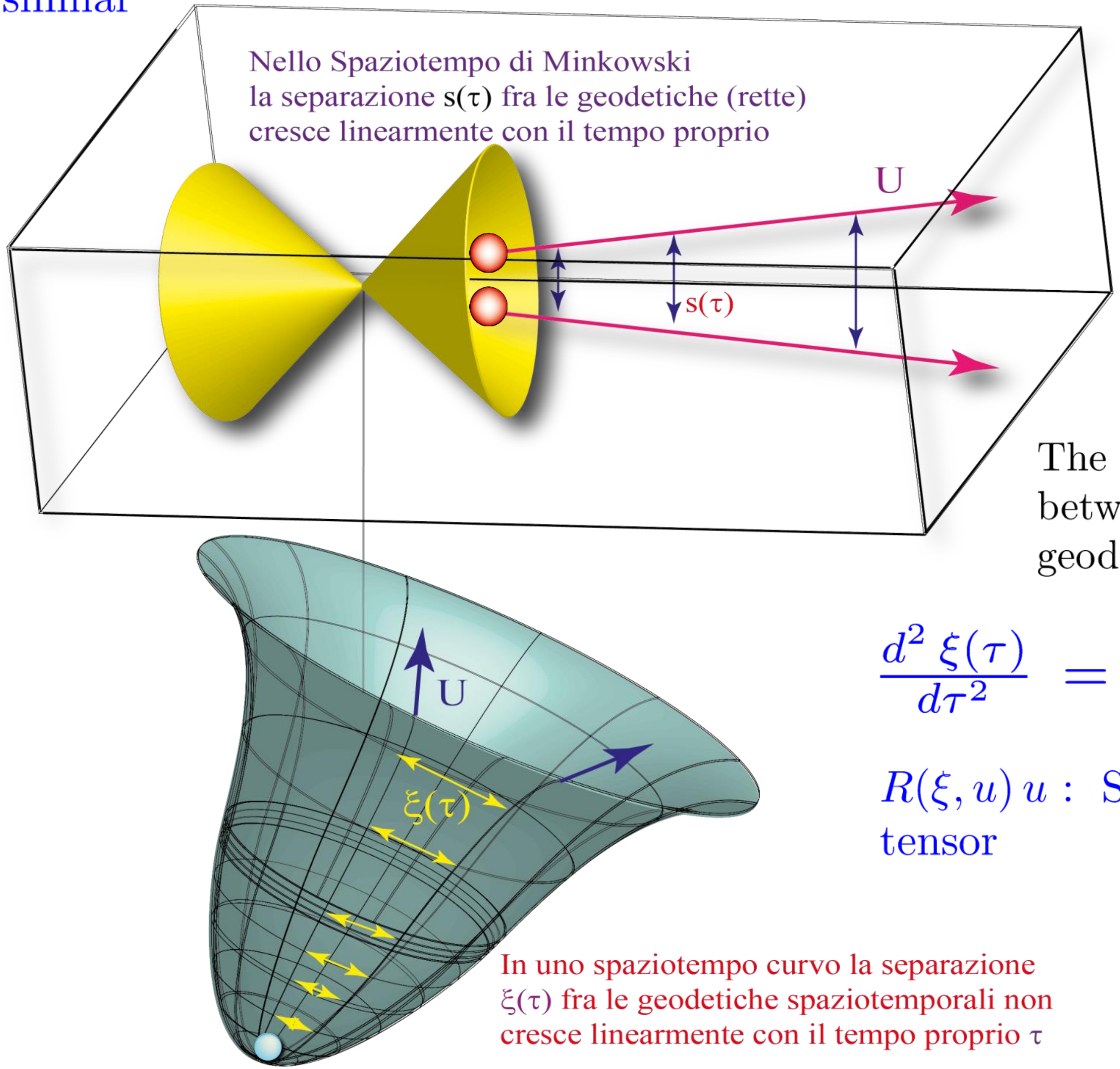
$$\frac{d\,\xi(x)}{dx} = \delta\phi \cos\left(\frac{x}{r}\right)$$

$$\frac{d^2\,\xi(x)}{dx^2} = -\frac{\delta\phi}{r} \sin\left(\frac{x}{r}\right)$$

$$= -\frac{1}{r^2}\,\xi(x)$$

$$= -\mathcal{K}_{Gauss}\,\xi(x)$$

The situation in a curved spacetime, as compared to the Riemannian case, is similar

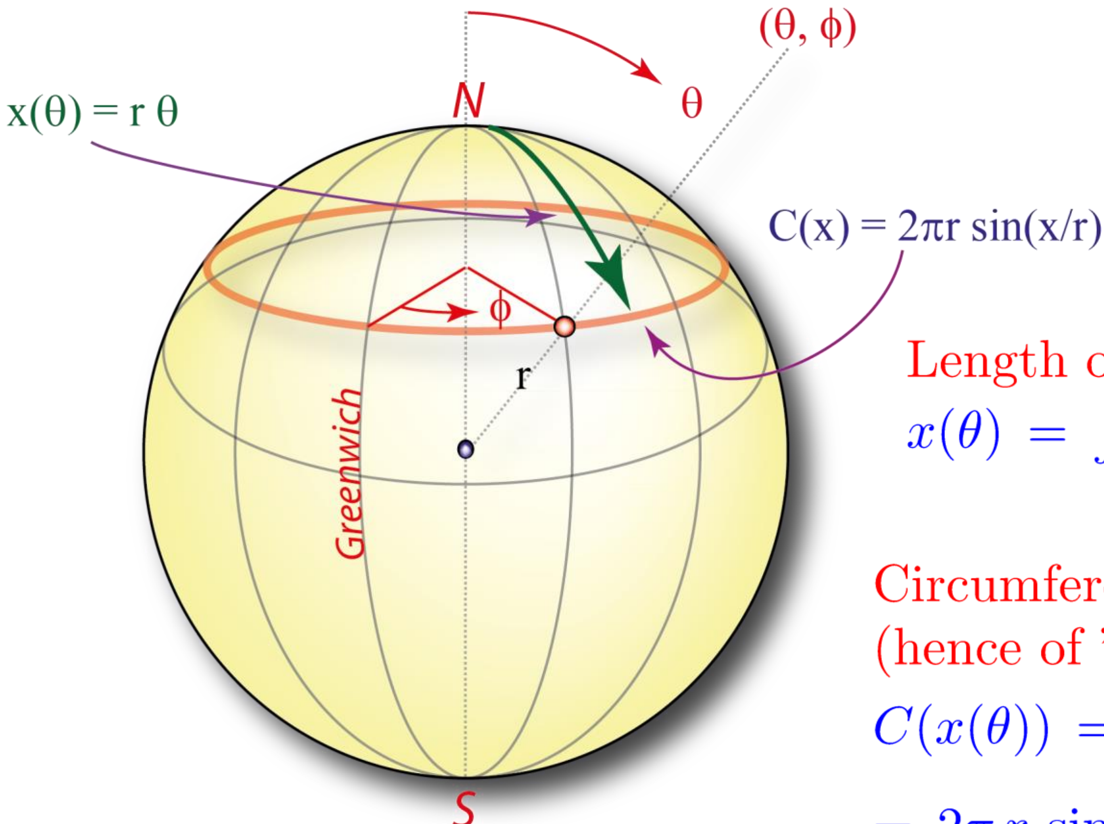


The relative acceleration between two close spacetime geodesics is given by

$$\frac{d^2\,\xi(\tau)}{d\tau^2} \;=\; -\,R(\xi,u)\,u$$

$R(\xi,u)\,u$: Spacetime Riemann tensor

Whereas is simple to understand how, from the metric $g = ds^2 = r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$ we can get infos on intrinsic geometry of the sphere:



Length of a meridian segment:

$$x(\theta) = \int_0^\theta ds|_{\phi=const.} = r \int_0^\theta d\theta' = r\theta.$$

Circumference of a parallel of latitude θ (hence of "radius" $x(\theta)$):

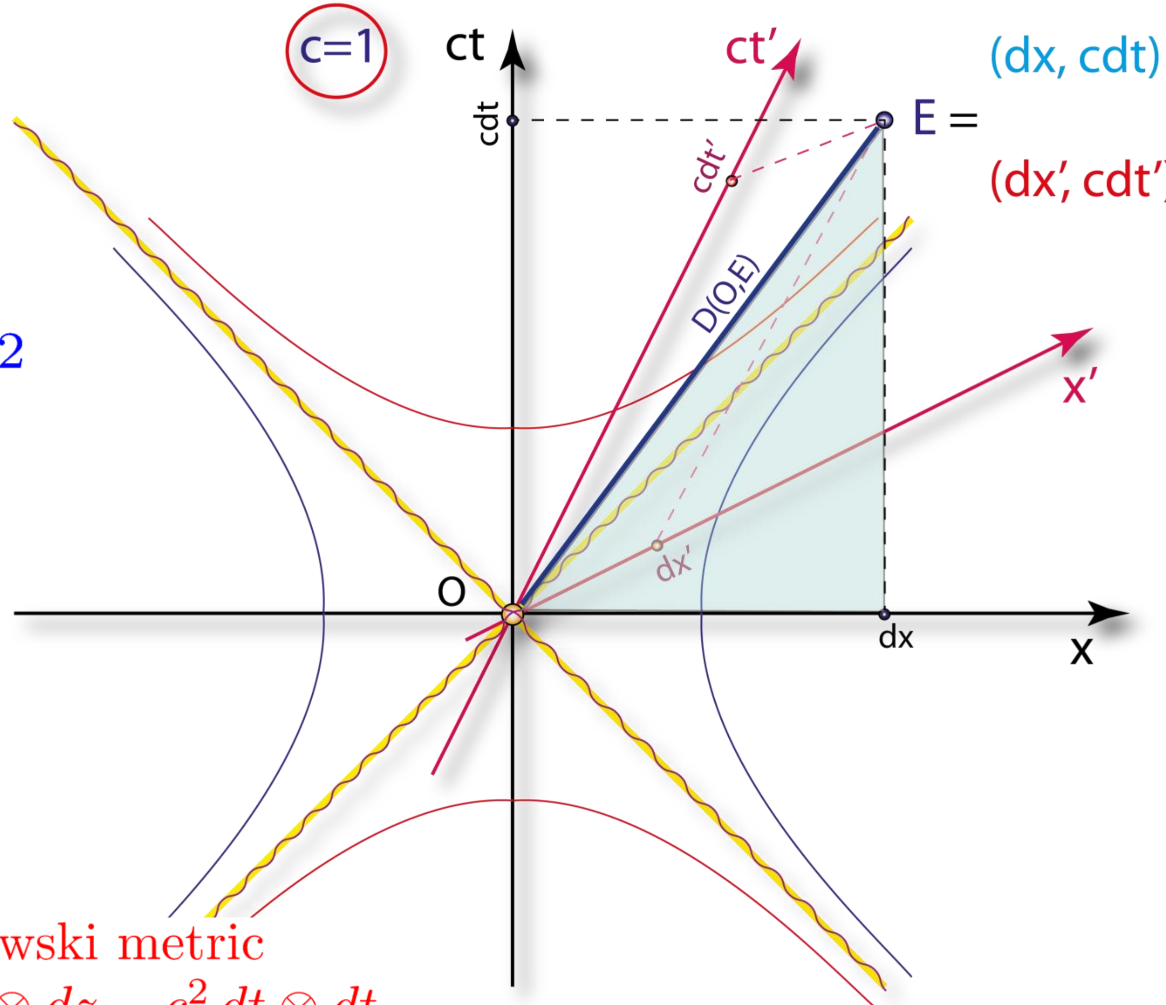
$$C(x(\theta)) = \int_0^{2\pi} ds|_\theta = r \sin\left(\frac{x(\theta)}{r}\right) \int_0^{2\pi} d\phi$$

$$= 2\pi \, r \sin\left(\frac{x(\theta)}{r}\right).$$

It is more difficult to understand what giving a spacetime metric means.

Recall that in Minkowski spacetime, the metric η is defined by the spacetime separation between the event $O = (0,0)$ and the infinitesimally close event $E = (dx, cdt)$. In dimension 2, for instance, we write

$$\eta := dx^2 - c^2\,dt^2$$
$$= dx'^2 - c^2\,dt'^2$$



In dimension 4 we get Minkowski metric
$\eta := dx \otimes dx + dy \otimes dy + dz \otimes dz - c^2\,dt \otimes dt$
often writen in an idiosyncratic notation as
$ds^2 = dx^2 + dy^2 + dz^2 - c^2\,dt^2$

Example: Let us assume that the spacetime metric is given by

$$g = "ds^2" = h(x,y,z,t)^2 \left(dx^2 + dy^2 + dz^2\right) \; - \; N^2(x,y,z,t)\, dt^2$$

How do we interpret it geometrically?

(i): The local geometry of time: Proper time along a spacetime curve at constant (x,y,z):

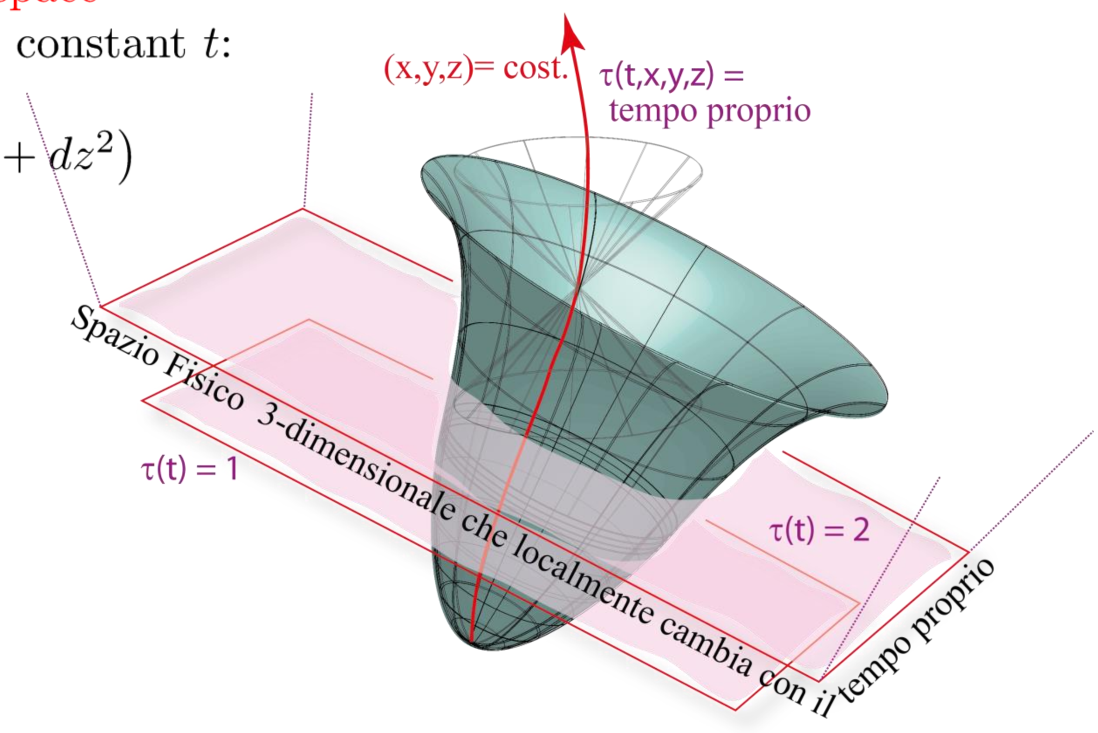
$$\tau(t;x,y,z) \; = \; \int_0^t \sqrt{-ds^2|_{(x,y,z)}} \; = \; \int_0^t N(x,y,z,t)\, dt'$$

(ii): The time–varying geometry of space

Metric of the 3-dimensional space at constant t:

$$ds^2|_{t=cost} \; = \; h(x,y,z,t)^2 \left(dx^2 + dy^2 + dz^2\right)$$

hence all distances in the 3-dim physical space locally change with (proper) time $t(\tau)$.



The Einstein equations connect the spacetime geometry to the distribution of mass–energy

How do we get them?

Whereas the constant gravitational fields are not "observable", the non uniform (tidal) fields are

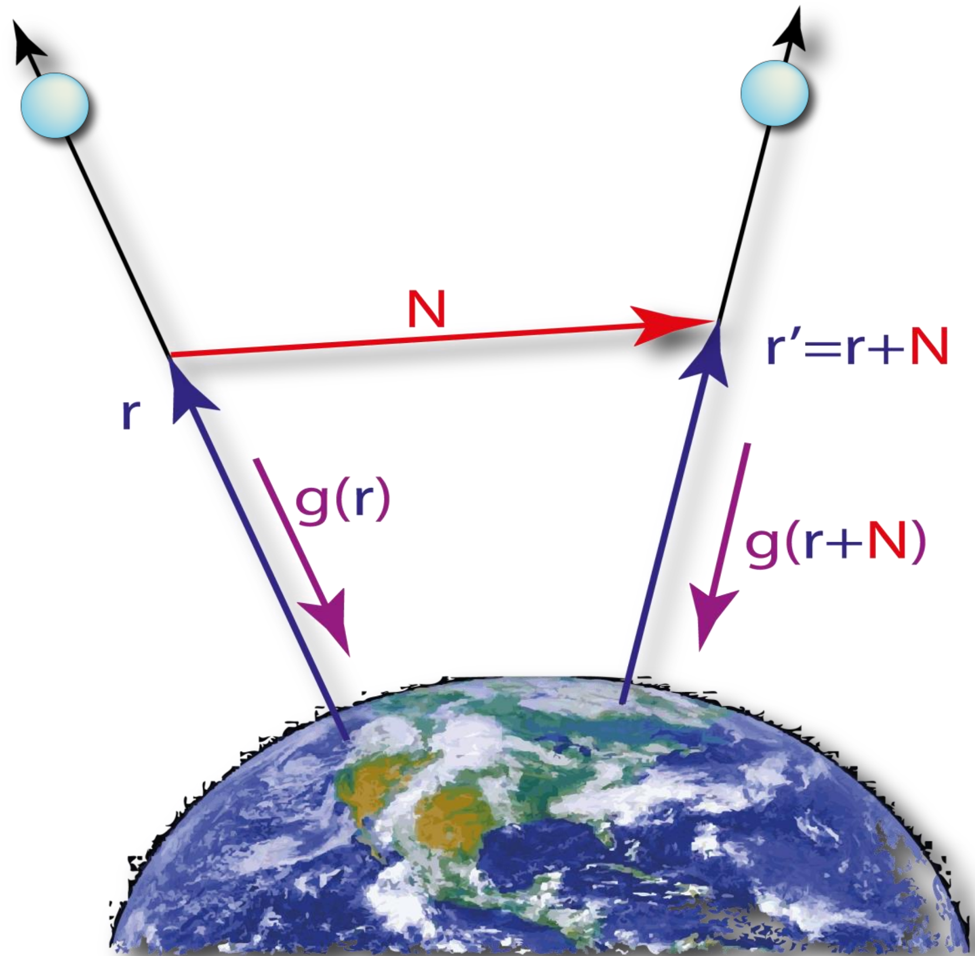
$$g(r) = -\,grad\,U$$

where U = gravitational potential

$$\frac{d^2 N^k}{dt^2} = -\,E^k_i\,N^i$$

where $E_{ik} = \frac{\partial^2 U}{\partial x^i\,\partial x^k}$

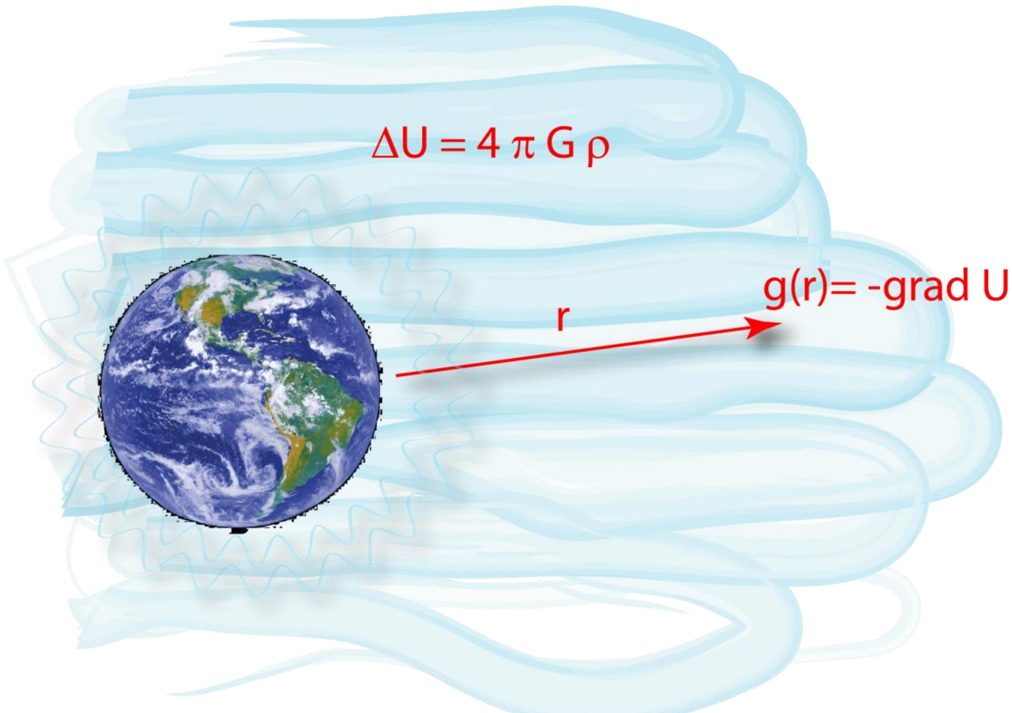
is the (Newtonian) tidal forces tensor.



In Newtonian theory, the connection between tidal forces and sources of the gravitational field is obtaines by "averaging" over the tidal forces along the distinct spatial directions:

$$E_{xx} + E_{yy} + E_{zz} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) U = 4\pi G \rho$$

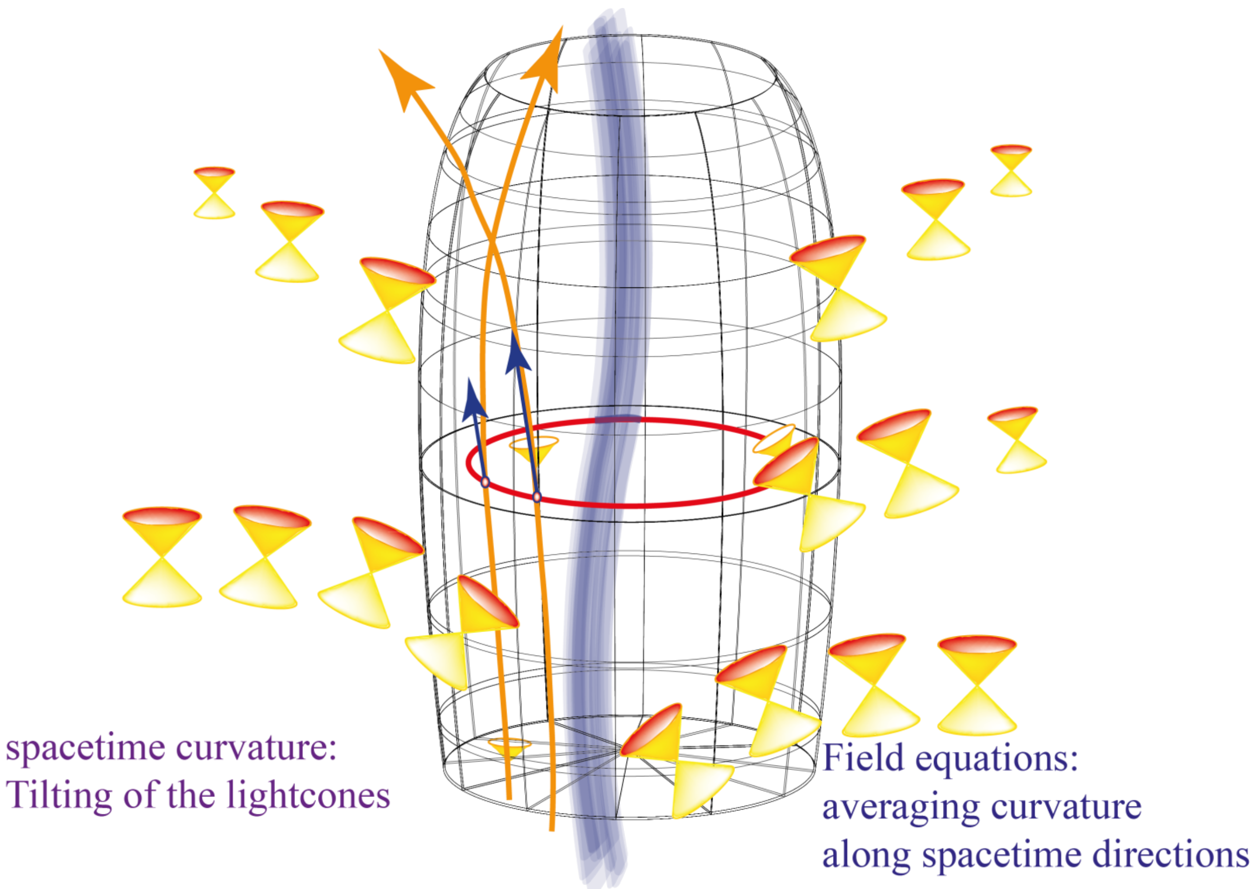
This is the Poisson equation connecting the Newtonian graitational field $g = -gradU$ to the matter density distribution ρ.



There is a natural analogy between the geodesic deviation equation describing the relative acceleration between geodesics due to the presence of curvature $R(\cdot,\cdot)\cdot$:

$$\frac{d^2\,\xi(\tau)}{d\tau^2} = -\,R(\xi,u)\,u \qquad\qquad \frac{d^2\,N^k}{dt^2} = -\,E^k_i\,N^i.$$

and the equation which describes the relative acceleration generated by tidal forces in Newtonian theory, represented by the tidal tensor $E(\cdot,\cdot)$



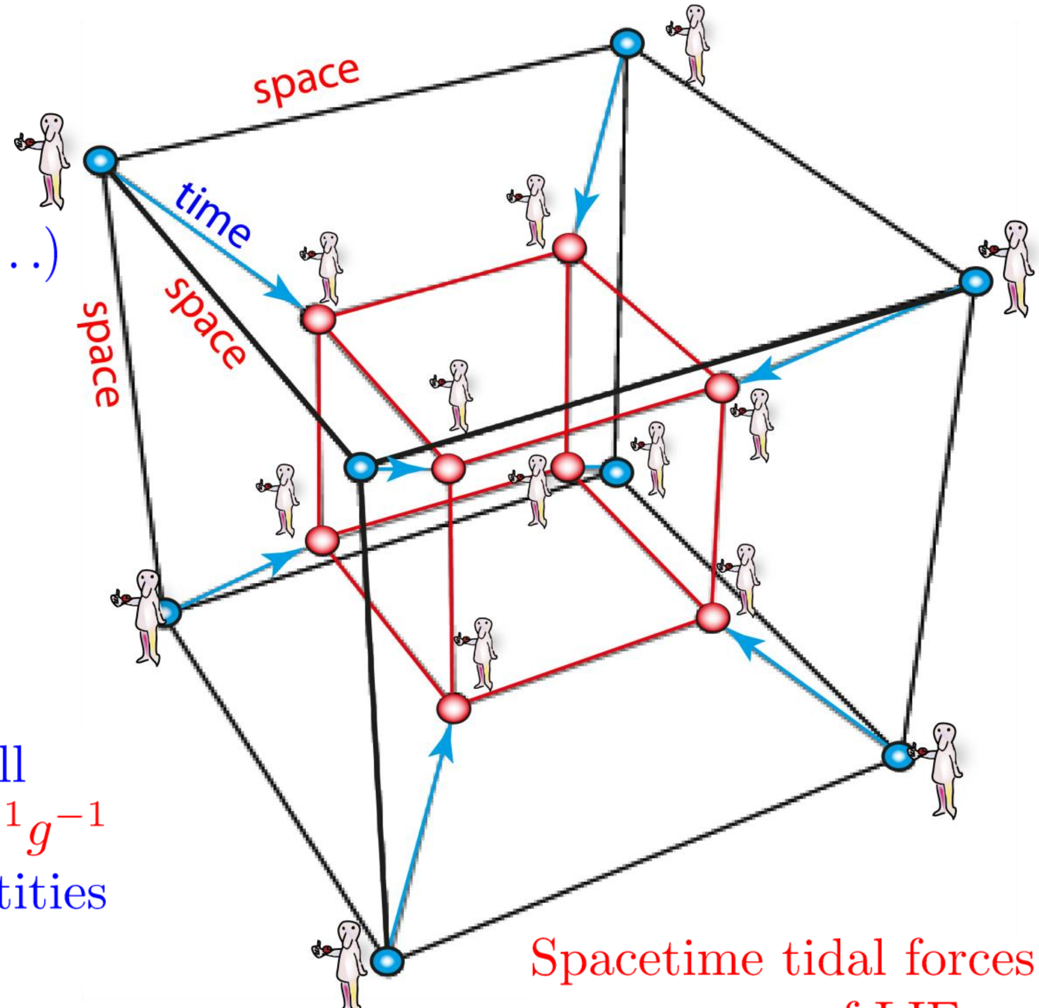
This analogy suggests that in the same way the average of $E(\cdot,\cdot)$ provides the connection with the sources of gravitational field in the Newtonian theory, a suitable "average" of the spacetime tidal deviations might provide the connection between spacetime curvature and mass–energy.

EINSTEIN EQUATIONS:

$$Ric(g) - \frac{1}{2}\, g\, R(g) = \frac{8\pi\, G}{c^4}\, T(g, \rho, p, \ldots)$$

where $T(g, \rho, p, \ldots)$ describes the distribution of mass–energy evolving in spacetime according to $\nabla^i\, T_{ik} = 0$.

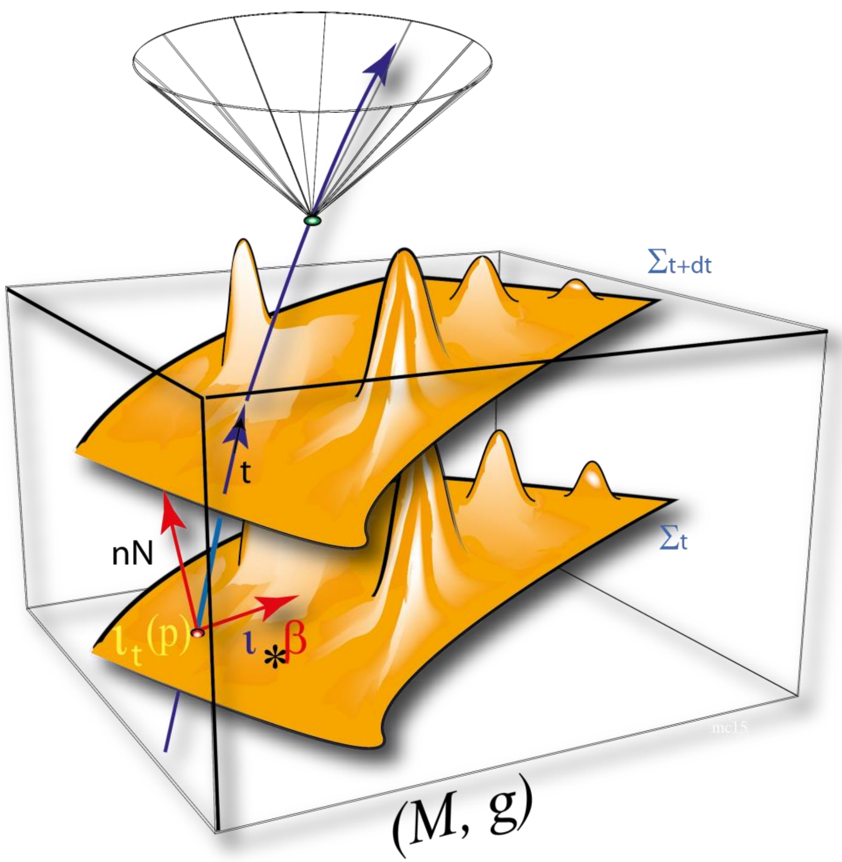
where $\frac{8\pi\, G}{c^4}$ has an exceedingly small numerical value $\simeq 2 \times 10^{-48}\, s^2 cm^{-1} g^{-1}$ if compared to other physical quantities occurring in the physics of sources.

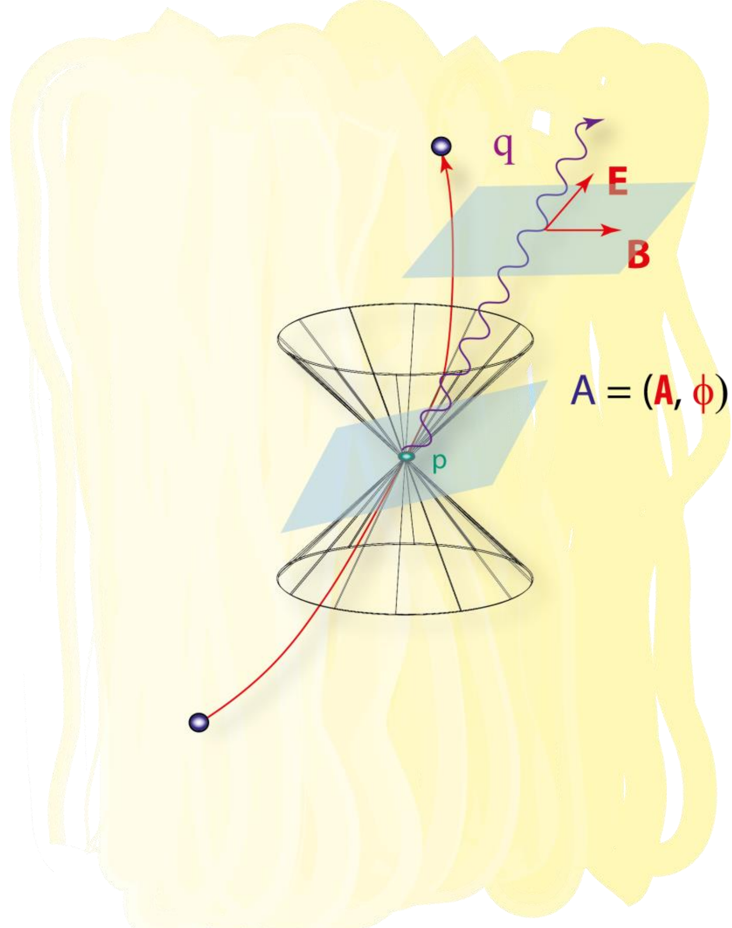


Spacetime tidal forces on an array of LIFs.

Understanding Einstein equations: Make comparison with Maxwell electromagnetism

- g_{ik}: 4–dimensional Spacetime metric $\Longrightarrow$ e.m 4–potential $A_h = (\vec{A}, \phi)$;
- $g^{(3)}_{ik}$: 3–dimensional metric of physical space $\Longrightarrow$ e.m vector potential $\vec{A}$;
- N: local geometry of time $\Longrightarrow$ e.m scalar potential ϕ;





MAXWELL EQUATIONS

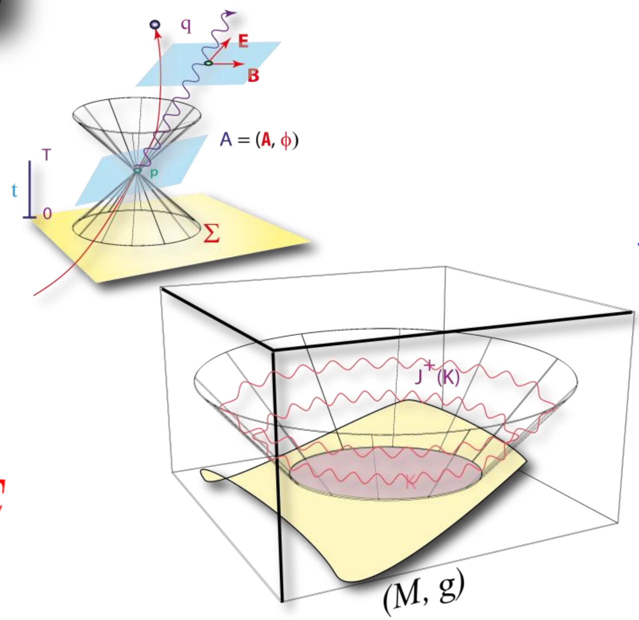
EINSTEIN EQUATIONS

$$div\, E \;=\; 4\pi\sigma$$

$$div\, B \;=\; 0$$

$$\frac{\partial B}{\partial ct} \;=\; -\; curl\, E$$

$$\frac{\partial E}{\partial ct} \;=\; curl\, B \;-\; \frac{4\pi}{c}\, j$$



$$\mathcal{R}^{(3)} + k^2 - K^a{}_b K^b{}_a = 16\pi\varrho$$

$$\nabla_b K^b{}_a - \nabla_a (tr_h K) = 8\pi J_a$$

$$\frac{\partial g^{(3)}_{ab}}{\partial t} = -2NK_{ab}$$

$$\frac{\partial K_{ab}}{\partial t} = -\nabla_a \nabla_b N + N\left[kK_{ab} + \mathcal{R}^{(3)}_{ab} - 2K_{ac}K^c_b\right]$$

As in any theory incorporating local Lorentz invariance, Einstein equations imply the existence of causal propagation of disturbances of the relevant fields, in this case of the spacetime metric (potential) and curvature (field): The Gravitational "Waves" .

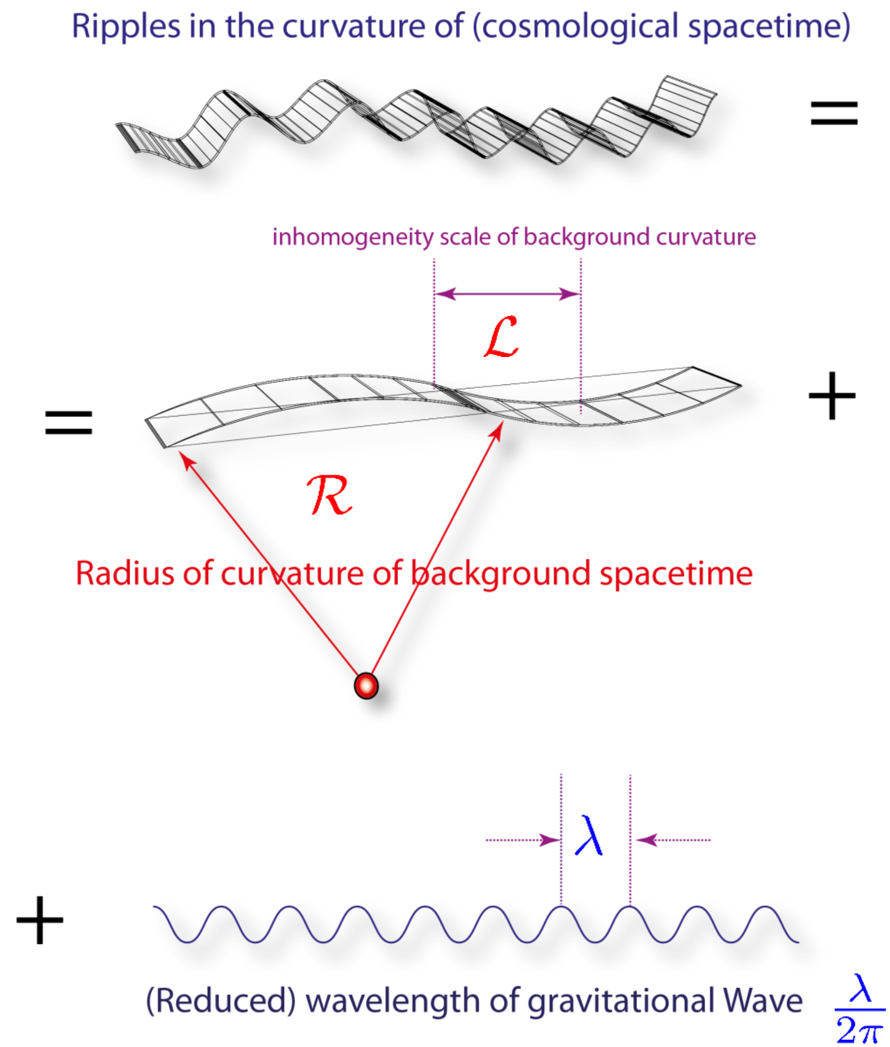
A Gravitational wave, (according to GR):

A Ripple in the curvature of spacetime propagating with the speed of light

Typically these ripples propagate on a background spacetime with a slowly changing spacetime curvature generated by the cosmological distribution of mass–energy.

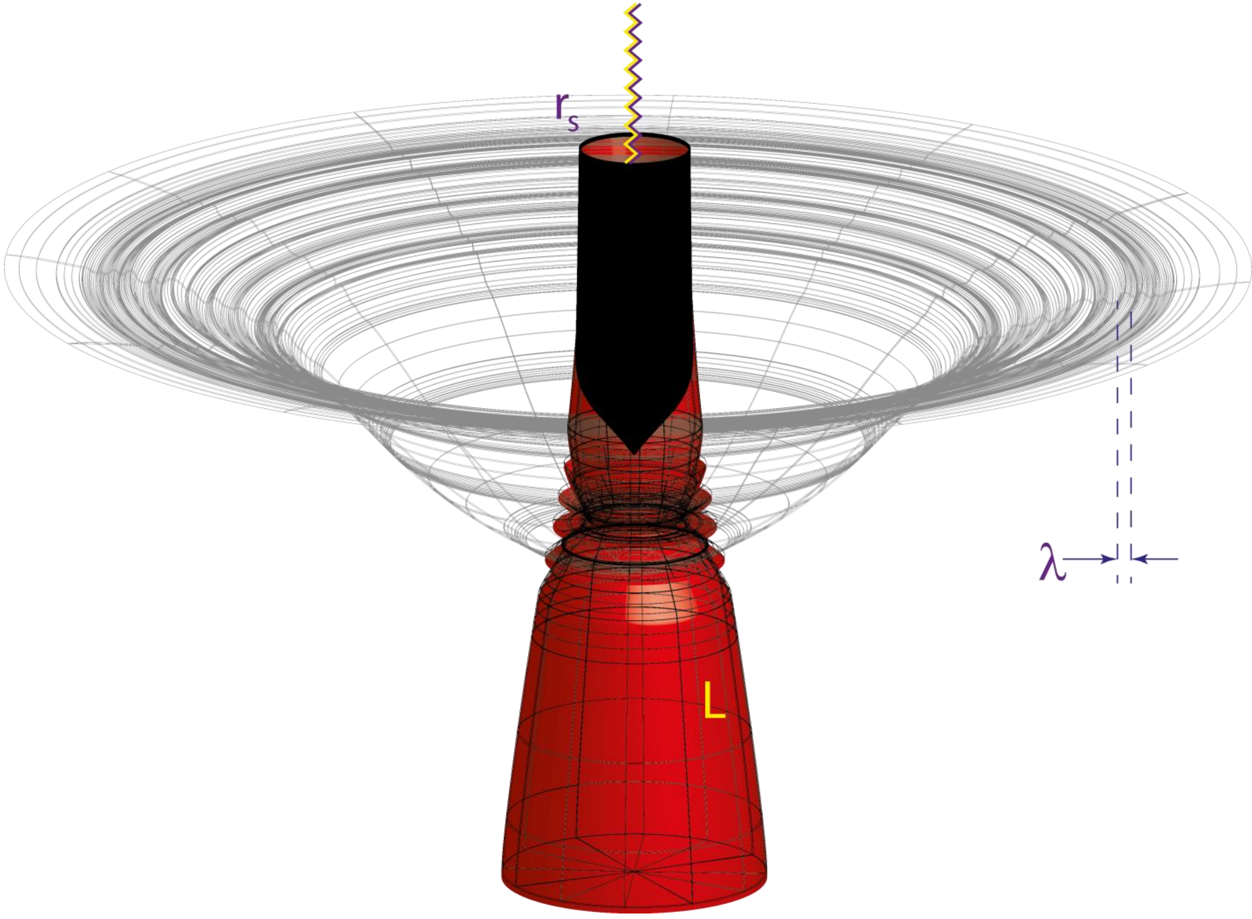
- The backgound curvature is characterized by two length scales: $\mathcal{R}$ and $\mathcal{L}$;
- Gravitational Waves are characterized by one length scale: $\frac{\lambda}{2\pi}$.
- The "separation" of spacetime curvature into a *Background* $Curv_{Back}$ and a *Wave Part* $Curv_{GW}$ depends critically on

$$\frac{\lambda}{2\pi} << \mathcal{L}$$



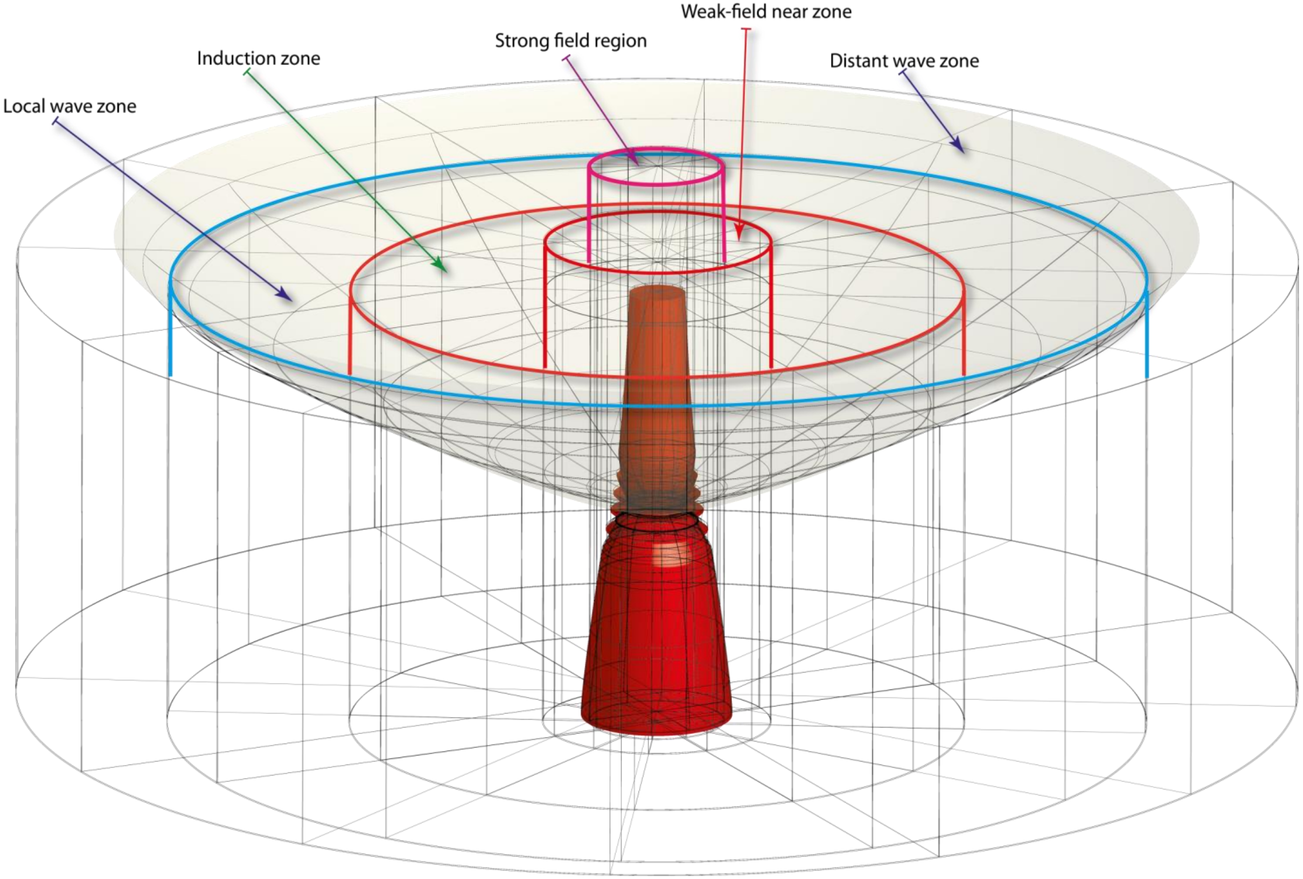
The length scales of a (potential) source of gravitational radiation

- L: The size of the source;
- r_s: Gravitational radius of the source, $(= 2M$, twice the "mass" of the source in geometrical units $G = 1 = c$. Explicitly $r_s = \frac{2GM}{c^2})$;
- $\frac{\lambda}{2\pi}$: Reduced wavelength of gravitational waves emitted by the source.



Regions of interest around a source of gravitational radiation

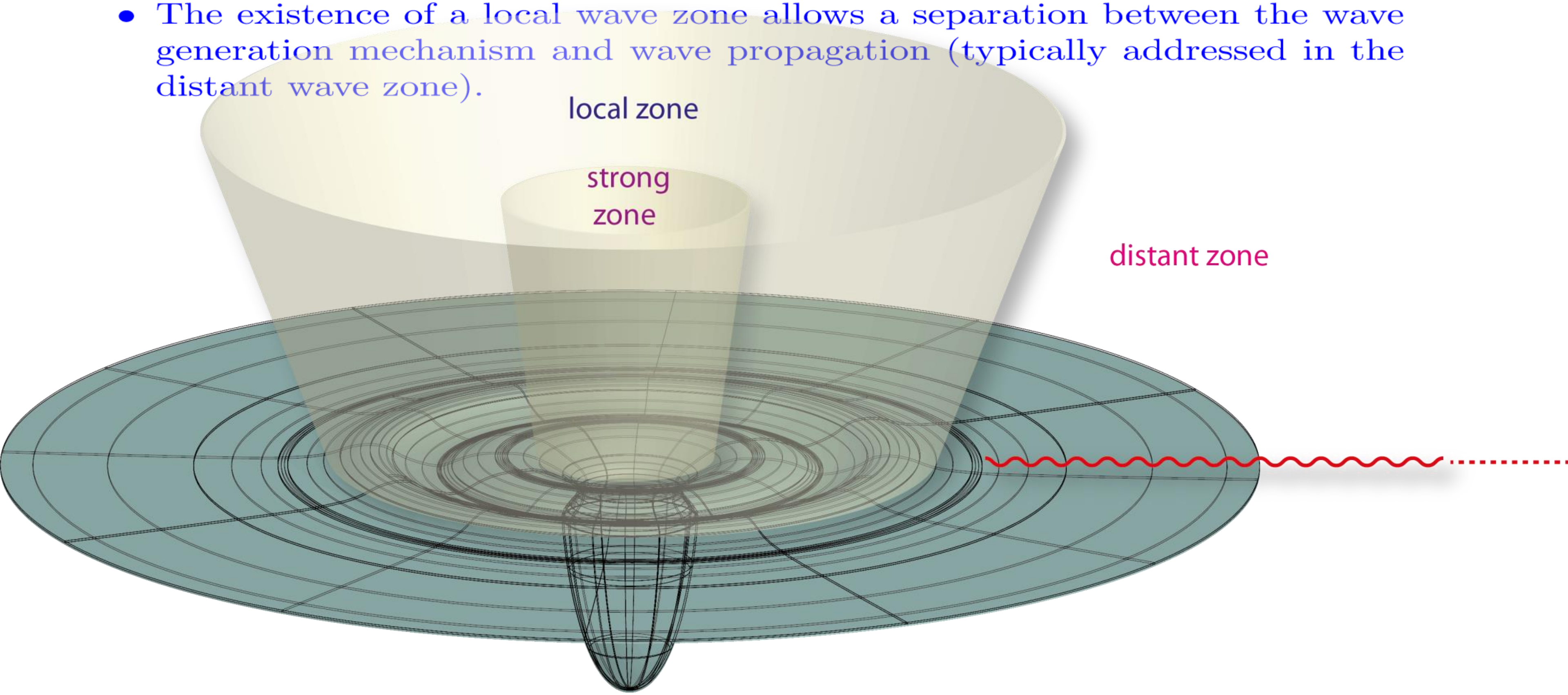
- Source region: $r \leq L$;
- Strong field region: $r \leq 5\,r_s$ (if strong field source, *i.e.* if $5\,r_s \geq L$);
- Weak field near zone: $L < r$, $5r_s << r$, $r << \frac{\lambda}{2\pi}$;
- Wave generation region(s): Source, Strong field region, weak–field near zone;



- Local wave zone: the region in which the GW generated by the source are weak outgoing ripples on a background spacetime and the effect of the background curvature on the wave propagation are negligible. Moving toward the wave generation region, the ripples cease to be waves and become near zone field ($r \leq \frac{\lambda}{2\pi}$). When $r \simeq r_s$ the source produces redshift and the background curvature distorts the wavefronts and backscatters the wave.

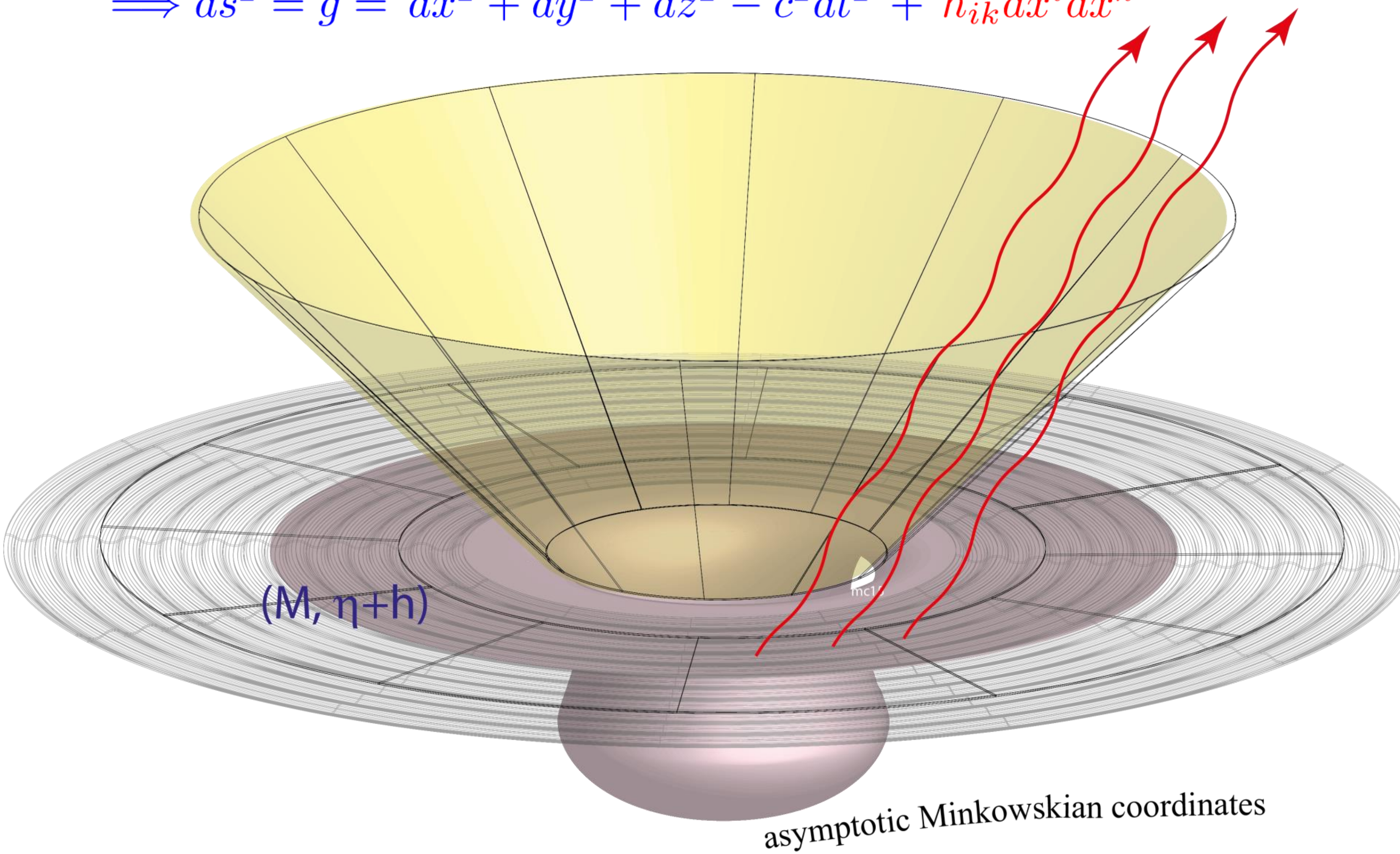
- Conversely when we move towards the distant wave zone there can be a phase shift build up generated by the gravitational field of the source and a potentially significant curvature background effect due to other sources (e.g. cosmological).

- The existence of a local wave zone allows a separation between the wave generation mechanism and wave propagation (typically addressed in the distant wave zone).



In the distant wave zone we can assume that the spacetime geometry is nearly Minkowskian: $g_{ik} = \eta_{ik} + h_{ik}$, where the (tensor) field h_{ik} can be thought of as a small perturbation of the Minkowskian spacetime geometry:

$$\Longrightarrow ds^2 = g = dx^2 + dy^2 + dz^2 - c^2 dt^2 + h_{ik} dx^i dx^k$$



Again, it is worthwhile to make a comparison with electromagnetism:

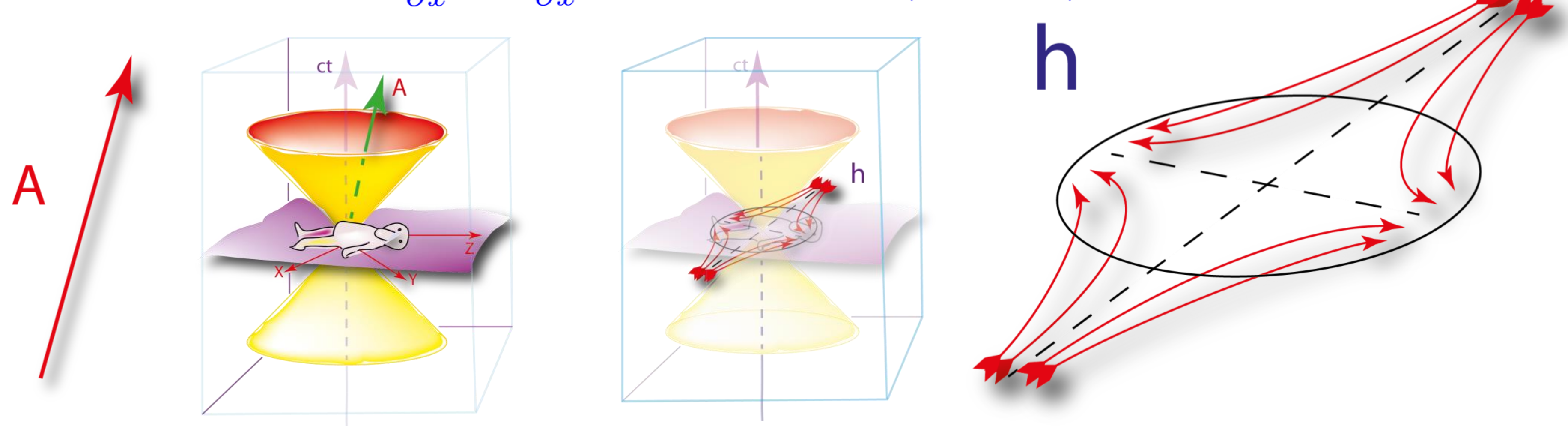
(i): The symmetric 2–tensor $h := h_{ik}dx^i dx^k$, describing gravitational perturbations, plays here the role that the 4–potential $A = (\vec{\mathbf{A}}, \phi)$ plays in electromagnetism.

(ii): Both are Lorentz equivariant quantities: $h = h_{ik}dx^i dx^k$ can indeed be interpreted as a symmetric 2–tensor in Minkowski spacetime; whereas $A = (\vec{\mathbf{A}}, \phi)$ is a Lorentz 4–vector.

(ii): Both are NOT uniquely defined: they are characterized up to a gauge trasformation:

$A_i \longmapsto \widetilde{A}_i = A_i + \frac{\partial \psi}{\partial x^i}$, where ψ is a scalar function.

$h_{ik} \longmapsto \widetilde{h}_{ik} = h_{ik} + \frac{\partial \xi_i}{\partial x^k} + \frac{\partial \xi_k}{\partial x^i}$, where ξ_k is a (Lorentz) vector field.



In electromagnetism, by imposing the Lorenz gauge (Ludvig Valentin L.!) $\frac{\partial}{\partial x^i} A^i = 0$, the Maxwell equations yields the wave equation

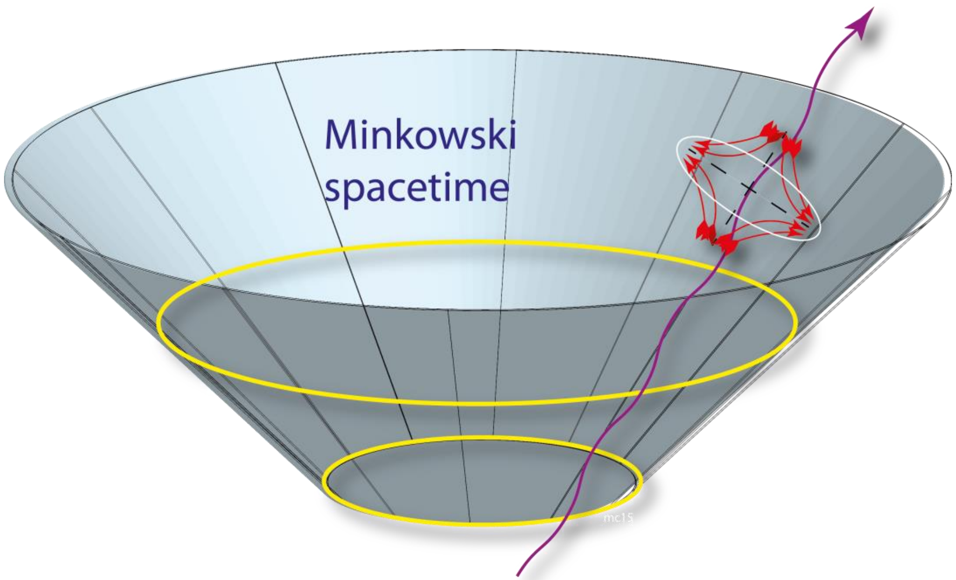
$$\Box_c A_i := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A_i = - \frac{4\pi}{c} J_i \ ,$$

Similarly, and quite remarkably, in the distant wave zone the Einstein equations, by imposing the gauge condition $\frac{\partial}{\partial x^i} h^{ik} = 0$, (the *harmonic gauge*), reduce to the standard wave equation

$$\Box_c \widehat{h}_{ik} := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \widehat{h}_{ik} = - \frac{16\pi\, G}{c^4} T_{ik} \ ,$$

where $\widehat{h}_{ik} := h_{ik} - \frac{1}{2} h\, \eta_{ik}$.

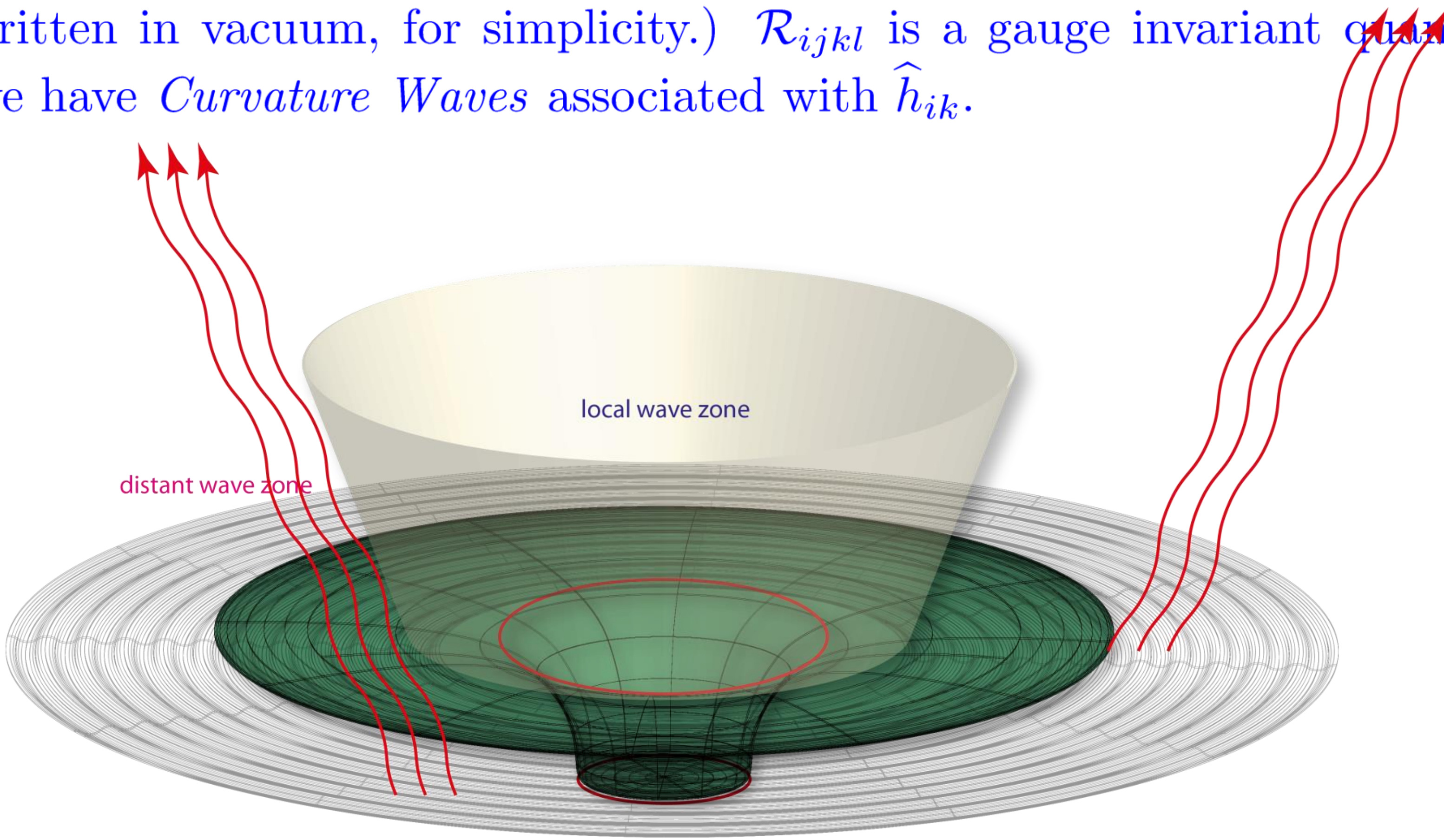
Hence, the tensor field $\widehat{h}$, describing the perturbations of the spacetime geometry in the distant wave zone can be interpreted as a free field (massless and with spin 2) evolving on a flat Minkowskian spacetime



Since the perturbation field h is NOT gauge invariant, there is the possibility that the solutions of the wave equation $\Box_c \widehat{h}_{ik} = -\frac{16\pi G}{c^4} T_{ik}$, (with $\partial_i \widehat{h}^{ik} = 0$), are just *Gauge Waves* of no physical relevance. However, along with the wave evolution of $\widehat{h}_{ik}$ we also get a wave equation for the associated curvature tensor

$$\Box_c \mathcal{R}_{ijkl} := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathcal{R}_{ijkl} = 0 \,, \qquad \Box$$

(here written in vacuum, for simplicity.) $\mathcal{R}_{ijkl}$ is a gauge invariant quantity, hence we have *Curvature Waves* associated with $\widehat{h}_{ik}$.



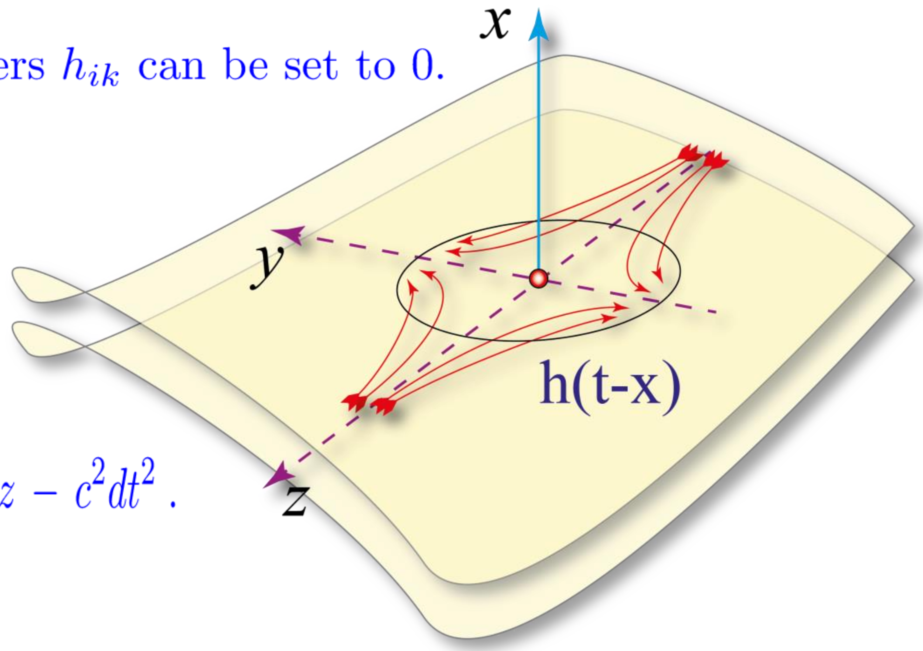
To understand the nature of $\widehat{h}$, let us consider a plane gravitational wave travelling in vacuum along the x–direction, (set $c = 1$) :

- Such a wave is described by the 10 functions $\widehat{h}_{ik}(t - x)$
- On them we have to impose the 4 conditions represented by the harmonic gauge $\frac{\partial}{\partial x^i} \widehat{h}_{ik} = 0$. This leaves us with 6 functions.
- We still have a residual gauge trasformation at our disposal, $h_{ik} \longmapsto h_{ik} + \partial_i \eta_k + \partial_k \eta_i$ where the vector field η_i satisfies the homogeneous wave equation $\Box_c \eta_i = 0$. Again 4 functions at our disposal to make an overall balance
- Hence, the physical information (the *Degrees of Freedom* describing the spacetime curvature wave field $\widehat{h}_{ik}$), is contained in just 2 functions:
- $h_{yy} = -h_{zz}$ and $h_{yz} = h_{zy}$, all others h_{ik} can be set to 0.

$$\mathcal{R}_{\alpha 11 \beta} = \mathcal{R}_{\alpha 44 \beta} = \frac{1}{2} \frac{d^2}{d(t-x)^2} h_{\alpha\beta}$$

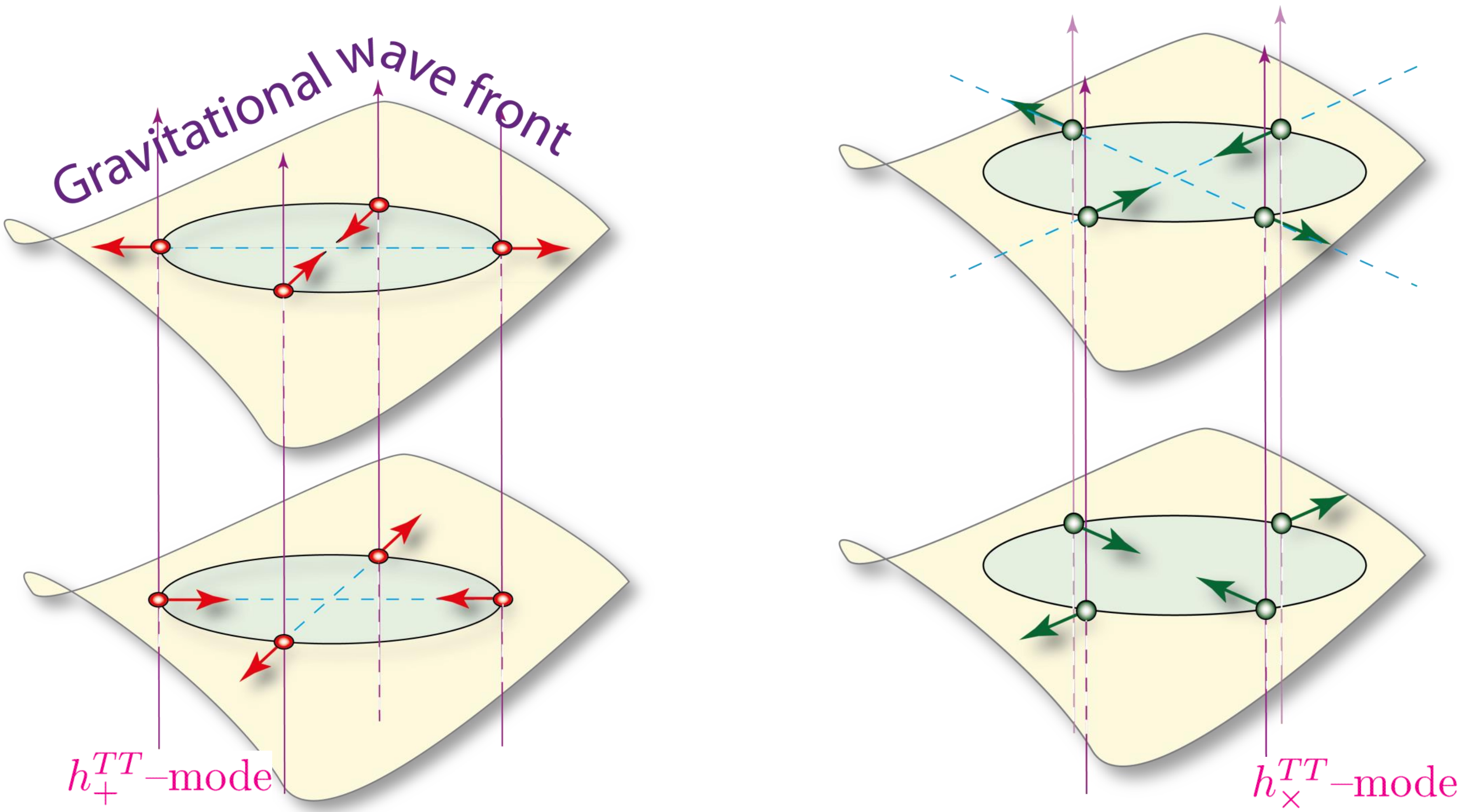
$$\mathcal{R}_{\alpha 14 \beta} = \mathcal{R}_{\alpha 41 \beta} = -\frac{1}{2} \frac{d^2}{d(t-x)^2} h_{\alpha\beta}$$

$$ds^2 = dx^2 + (1 - h_{yy})dy^2 + (1 + h_{yy})dz^2 - 2h_{yz}dydz - c^2dt^2 .$$

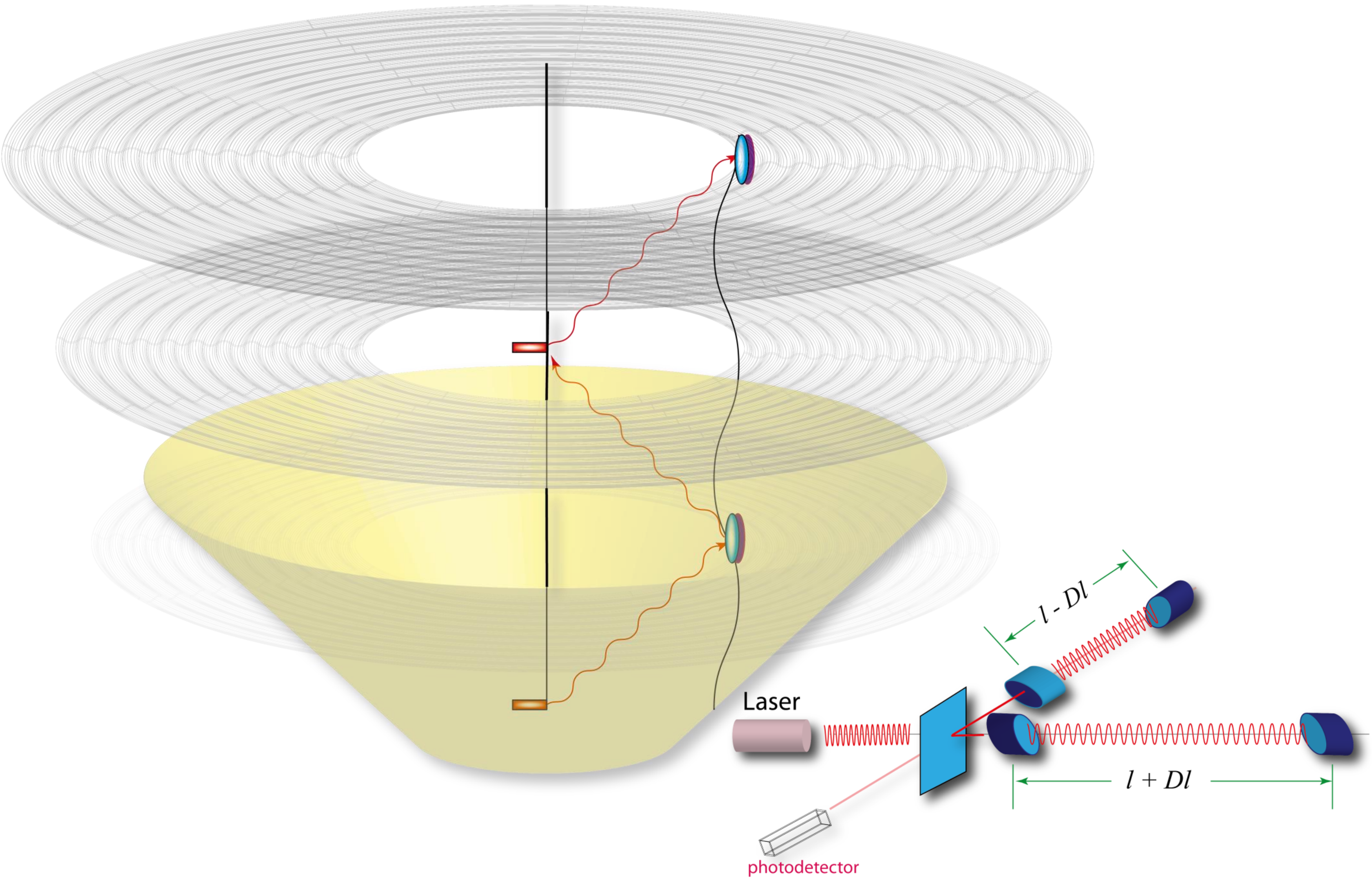


The relevant part of $\widehat{h}$, describing the spacetime curvature waves, is the trace-free and divergence–free part: h^{TT}_{ab} such that $\eta^{ab}\, h^{TT}_{ab} = 0$, and $\eta^{ab} \frac{\partial}{\partial x^b} h^{TT}_{ak} = 0$.

The two independent fields h^{TT} give rise to mareal forces on a cloud of free falling matter test particles according to the following two–polarizations pattern:

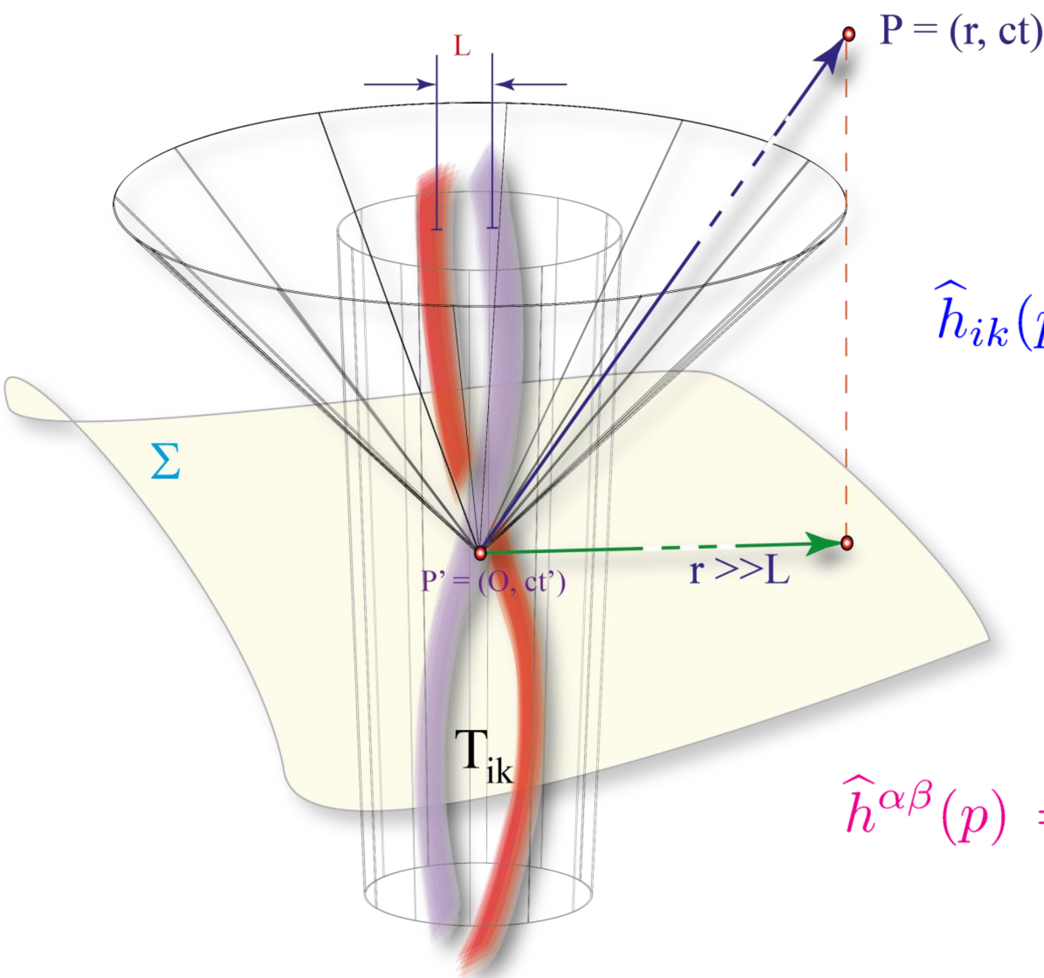


The h^{TT}–induced relative mareal displacement $2\frac{\Delta l}{l}$ on free–falling test particles can be measured by interferometric techniques.



But how large the h^{TT} mareal relative displacement $2\frac{\Delta l}{l}$ can be?
To answer we need to look into the mechanism of gravitational wave generation, (in the weak–field case!)

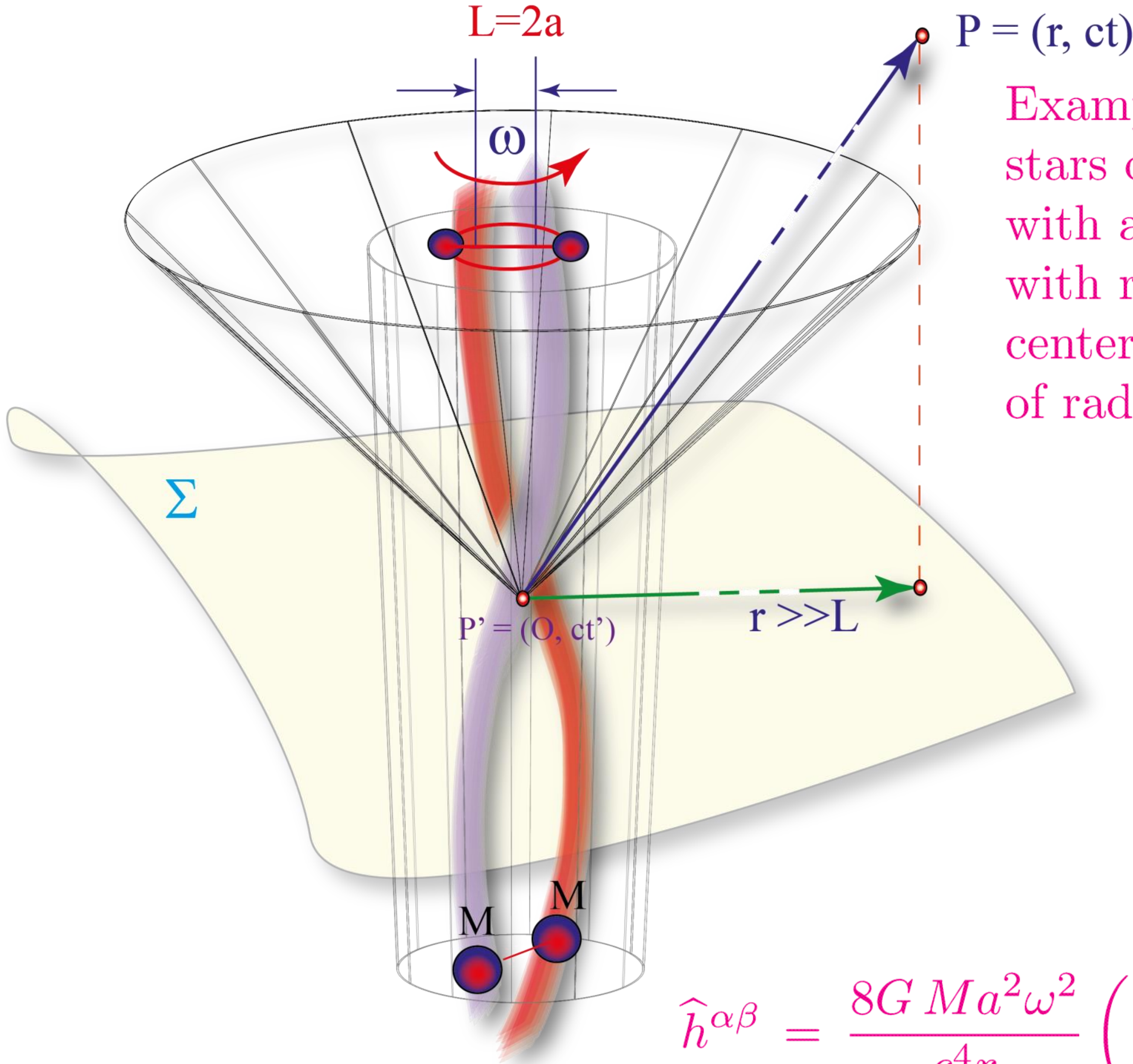
As in electrodynamics, the wave equation $\Box_c \widehat{h}_{ik} = -\frac{16\pi G}{c^4} T_{ik}$ can be explicitly solved as a retarded potential problem:



$$\widehat{h}_{ik}(p) = -\frac{4G}{c^4} \int_{\Sigma_{ret}} \frac{[T_{ik}]_{ret}}{r} dV^{(3)}$$

The Quadrupole:

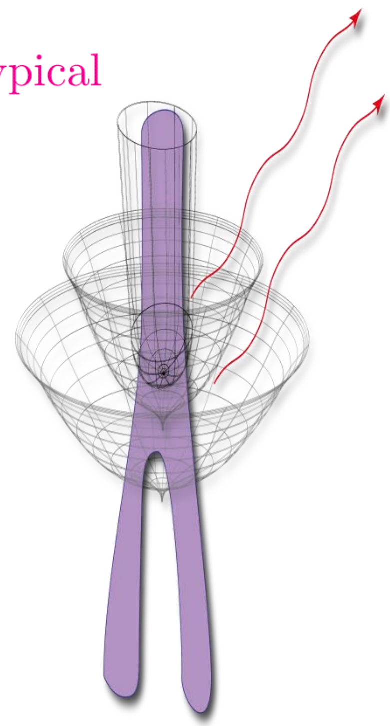
$$\widehat{h}^{\alpha\beta}(p) = -\frac{2G}{c^4 r} \frac{d^2}{dt^2} \int_{\Sigma} \rho\, x^\alpha x^\beta \, dV^{(3)}$$



Example: A system of two stars of mass M rotating, with angular velocity ω, with respect to a common center, on a circular orbit of radius $2a$.

$$\widehat{h}^{\alpha\beta} = \frac{8G\,Ma^2\omega^2}{c^4 r}\begin{pmatrix} \cos 2\omega t & \sin 2\omega t \\ \sin 2\omega t & -\cos 2\omega t \end{pmatrix}$$

The smallness of the numerical factor $\frac{8G\,Ma^2\omega^2}{c^4 r}$ is scaring!



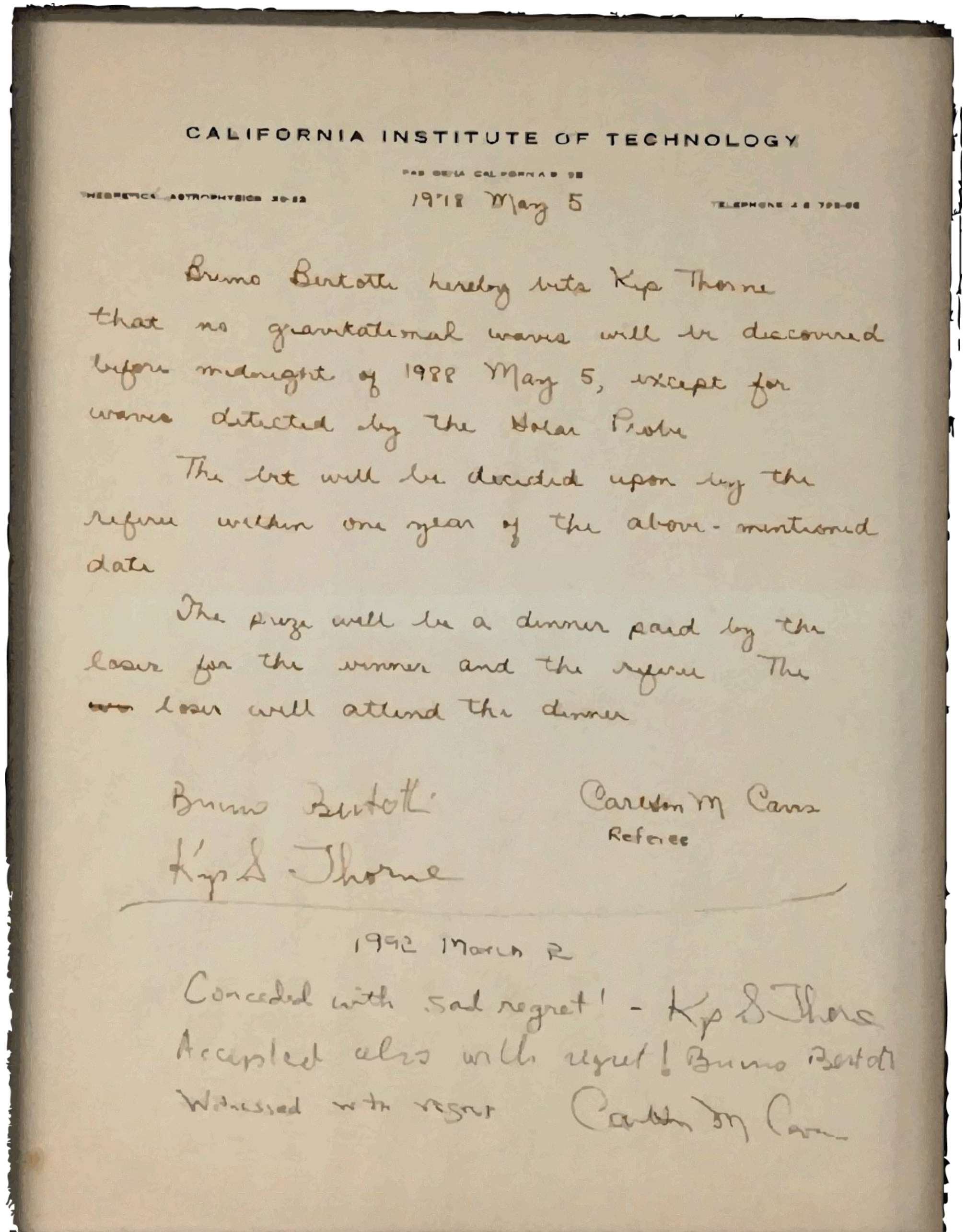
- The number: Typical masses of a neutron star ($1.4\,M_\odot$) at the typical distance of the Virgo Cluster of Galaxies (50 Mpc, $i.e.\, r \simeq 4.6 \times 10^{23}\,m$), yields a gravitatational wave amplitude $2\frac{\Delta l}{l}$ of the order of 10^{-20}.

- But Beware! The energy carried by the waves can be large: A typical formula for the Energy Flux, *i.e.* the energy carried by a gravitational wave, of amplitude h and frequency ν, through a unit area per unit time is

$$\mathcal{F} \simeq \frac{\pi c^3}{4G}\,\nu^2\,h^2.$$

- However! The energy of a GW wave is *well defined* only as an average over a region of space with size (quite) larger than the wavelength of the wave, and over a time larger than the period of the wave. Difficult to transfer significant GW energy to matter and to an antenna!

Hence, Bruno had good reasons to propose to Kip Thorne the following bet:



CALIFORNIA INSTITUTE OF TECHNOLOGY

1978 May 5

Bruno Bertotti hereby bets Kip Thorne that no gravitational waves will be discovered before midnight of 1988 May 5, except for waves detected by the Solar Probe.

The bet will be decided upon by the referee within one year of the above-mentioned date.

The prize will be a dinner paid by the loser for the winner and the referee. The ~~wo~~ loser will attend the dinner.

Bruno Bertotti

Carleton M Caves
Referee

Kip S Thorne

1992 March 2

Conceded with sad regret! - Kip S Thorne

Accepted also with regret! Bruno Bertotti

Witnessed with regret Carleton M Caves