UNIVERSALITY, CONDENSED MATTER AND QUANTUM FIELD THEORY

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I. UNIVERSALITY

Matter is composed by atoms or molecules mutually interacting and obeying to the laws of quantum mechanics; the delicate interplay of quantum mechanical effects and collective phenomena related to the enormous number of particles is at the origin of the remarkable properties exhibited at low temperatures by several materials, superconductivity being a classical example.

The structure of matter is extremely complex and depending on an enormous number of parameters, and one should expect that this is reflected by a very complicated dependence of the macroscopic observables on such parameters. It is an experimental fact, on the contrary, that certain quantities exhibit independence from the underlying structure, having the same value in large classes of systems with different microscopic composition. This *universal* behavior suggests that certain macroscopic properties are sensitive more to abstract mathematical structures than to the microscopic details, and can therefore be the same also in the idealized models accessible to theoretical analysis. This opens the possibility of exact quantitative predictions for real systems starting from ideal or very approximate models. The situation recalls the emergence of gaussian law in central limits in probability, where a similar independence from details is found in a much simpler setting; remarkably, universality in condensed matter regards even the precise value of certain physical observables and not simply distribution laws.

A classical example of universality appears in the second order phase transitions occurring at a certain critical temperature, where several physical properties have a non-analytic behavior driven by a *critical index*. While the critical temperature depends on microscopic details, the exponents are typically universal; for instance [1] the value of the index β at the ferromagnetic transition is 0.119(8) in Rb_2CoF_4, 0.123(8) in K_2CoF_4, 0.135(3) in Ba_2FeF_6. Despite such materials are composed by very different molecules, the exponents are essentially identical (up to small computational or experimental errors); in addition, even if the Ising model in 2 dimensions is rather idealized and has only a vague resemblance with the microscopic structure of such materials, it has essentially the same exponent

1/8. This universal behavior is a collective effect due to the interaction of a large number of particles, and has found an explanation in the Renormalization Group approach, developed starting from the deep work of Kadanoff [2] and Wilson [3].

While in the above example universality is rather independent from the classical or quantum nature of particles, in other cases it is a truly quantum phenomenon. This is the case of the quantization of the Hall conductivity, see Fig.1, which is expressed by an universal constant e^2/h times an integer or fractional number; here one has not only perfect independence from all the microscopic parameters but also a quantization phenomenon. The explanation has been found, at least neglecting the effect of the interaction, identifying the Hall conductance with a topological invariant [4],[5]: the first Chern number of a certain bundle associated with the ground state of the quantum Hamiltonian. Quantized and universal conductance is observed in Chern or topological insulators [6].

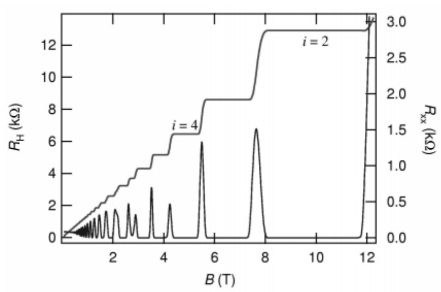

Figure 1: Quantization of the Hall conductivity

A more recent example of universal conductivity has been found in graphene, the newly discovered material [7] composed by a monoatomic crystal of carbon atoms. An experiment [8], see Fig. 2, shows that the optical longitudinal conductivity is equal to $\sigma_0 = \frac{e^2}{h}\frac{\pi}{2}$, an expression obtained by a very idealized model of non interacting electrons on the honeycomb lattice. This agreement is again a striking example of universality as the charge carriers in graphene are surely

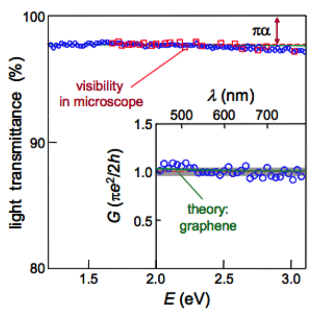

Figure 2: The optical conductivity in graphene

strongly interacting via Coulomb forces, and many body effects are experimentally seen in several physical properties, like the Fermi velocity, but not on the conductivity. A similar universality phenomenon is present also in the Hall conductivity: experiments do not show any correction due to the interaction, at least if weak enough.

There are also subtler forms of universality; in certain cases the observables do depend on microscopic details but there exist suitable relations between them which are universal. This is what happens in a class of one dimensional conductors named by Haldane [9] Luttinger liquids; their conductivity, exponents and susceptibility depend on microscopic details but verify universal relations, allowing for instance to predict the exponents once that the conductivity and susceptibility are known. One dimensional fermionic systems are not easily realized in nature, and among their clearer realization are the edge states of 2d topological insulators

The understanding of the universality phenomena like the ones briefly mentioned above (and many others) have stimulated an enormous theoretical activity and the explanations require a combination of physical intuition with surprisingly abstract mathematical developments. In addition, it turns also out that universality has a quite strict relations with other deep phenomena happening in apparently unrelated fields, like Quantum Field Theory (QFT); even if QFT describes quantum relativistic particles at extreme energies and the microscopic components of matter obey to non relativistic quantum mechanics, several universal phenomena have a common mathematical origin. Despite enormous progresses, we are still at the beginning and many basic questions and observations need an explanation.

II. HONEYCOMBS

A major role in our understanding is played by simple but extremely deep models, accessible to exact or rigorous analysis, the classical example being provided by the Ising model. More recently, a system that had a major role in many body theory is given by a tight binding model of fermions hopping in the honeycomb lattice, see Fig.3. The model was proposed in [10]; if $\Lambda = \Lambda_A \bigcup \Lambda_B$ is the honeycomb lattice, $x + \delta_i$ the n.n. sites to x, and $a^{\pm}, b^{\pm}$ are fermionic creation or annihilation operators, the Hamiltonian in second quantization is, at half filling

$$H_0 = -t \sum_{\vec{x} \in \Lambda_A, i=1,2,3} \sum_{\sigma = \uparrow\downarrow} \left(a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_i,\sigma} + b^+_{\vec{x}+\vec{\delta}_i,\sigma} a^-_{\vec{x},\sigma} \right)$$

This system was studied at the beginning essentially for its theoretical interest, as at low energies it admits an effective description in terms of massless Dirac

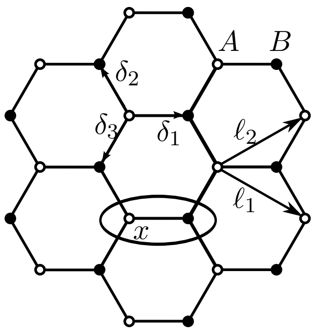

Figure 3: The honeycomb lattice Λ

fermions in $2+1$ dimensions, and this suggests the possibility of observing in condensed matter the analogue of QFT phenomena. Far from a mere theoretical curiosity, this model has provided later a quite accurate description of the charge carriers of graphene; the theoretical computation of the optical conductivity in Fig.2 is obtained using this simple model. Note that in the above model the only microscopic parameter is the hopping t, but the conductivity is independent from it. It is of course a simplification of a more realistic system describing quantum particles in the continuum with a periodic potential with honeycomb symmetry [11].

Subsequently, Haldane [12] provided another key step adding an extra interaction with next to neighbor sites, that is $H_0 + H_1$ where

$$H_1 = -t_2 \sum_{\alpha=\pm} \left(e^{i\alpha\phi} a^+_{\vec{x},\sigma} a^-_{\vec{x}+\alpha\vec{\ell}_j,\sigma} + e^{-i\alpha\phi} b^+_{\vec{x}+\vec{\delta}_1,\sigma} b^-_{\vec{x}+\vec{\delta}_1+\alpha\vec{\ell}_j,\sigma} \right) + \frac{M}{3} \left(a^+_{\vec{x},\sigma} a^-_{\vec{x},\sigma} - b^+_{\vec{x}+\vec{\delta}_j,\sigma} b^-_{\vec{x}+\vec{\delta}_j,\sigma} \right) .$$

This model is the prototype of the Chern insulators. The term H_1 induces a gap in the spectrum so that if the chemical potential is in the gap the system is insulating, that is the longitudinal conductivity is vanishing. However, the system has a transversal Hall conductivity which is universal (that is, independent from t, M, ϕ) and perfectly quantized, which can be expressed in terms of a Chern number; that is it exhibits a non trivial Hall effect without a net external magnetic field. In Fig.4 it is showed the region in parameters space where the transversal equal to the value $\pm 2e^2/h$; at the origin one recovers instead the value of the optical conductivity of graphene $\frac{e^2}{h}\frac{\pi}{2}$. Universality also persists adding a stochastic term describing disorder, as a consequence of the topological interpretation [13],[14]. The Haldane model has been experimentally realized in cold atoms experiments [15]. In the gapped region the longitudinal conductivity is vanishing, but the system can still support non vanishing currents at the surface [9]; remarkably the edge conductance is equal to the transversal conductivity, and therefore quantized and universal. This is a manifestation of a rather general fact known as bulk-edge correspondence which in this model can be explicitly

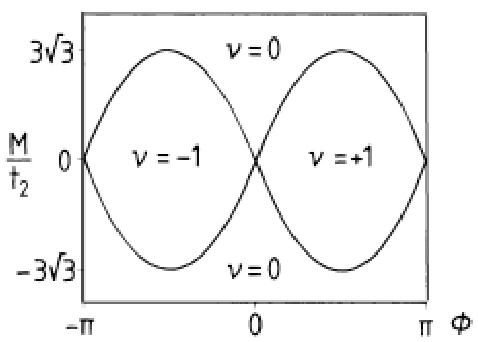

Figure 4: Phase diagram of the Haldane model

verified. Note also that the electrons circulating at the surface provide a physical realization of a one dimensional electronic systems.

Finally Kane and Mele [16] considered the sum of two Haldane models describing particles with opposite spin, with phase ϕ and $-\phi$ respectively; this model is the prototype of topological insulator and has a quantized universal spin Hall conductivity without breaking time reversal.

Therefore, the simple model of electrons on the honeycomb lattice is a prototype for several universality phenomena in condensed matter, ranging from graphene to topological or Chern insulators, and, via the bulk-edge correspondence, to one dimensional electron systems. The absence of interaction makes the model exactly solvable, and its properties fully determined by the single particle Schroedinger equation.

Experiments, however, refer to real materials in which interactions cannot be neglected and is often quite strong, and one needs to investigate the interplay of interactions and universality in the interacting versions of models of fermions on the honeycomb lattice. The role of interaction in the above universality phenomena is a major question. In the case of weak interactions, some progress has been recently achieved, which will be shortly reviewed in the rest of this note.

III. CONDENSED MATTER AND QFT

An interacting version of systems of electrons on the honeycomb lattice is obtained for instance by adding to the Hamiltonian H_0 or $H_0 + H_1$ a term of the form

$$V = U \sum_{\vec{x}} \left(n_{\vec{x},\uparrow} - \frac{1}{2} \right) \left(n_{\vec{x},\downarrow} - \frac{1}{2} \right) ,$$

where $n_{\vec{x},\sigma}$ is the fermionic density, obtaining respectively the graphene-Hubbard or Haldane-Hubbard model. This term describes a density-density interaction with coupling U, like a screened Coulomb potential or more generically some effective force mediated by the lattice. It is an on-site ultralocal interaction but one could consider also nearest-neighbor or short range interaction, and no qualitative difference are expected in the weak coupling regime; in contrast long range interactions, like the Coulomb one, are expected to produce a radically different behavior.

A major difficulty is that, in presence of interaction, the physical properties cannot be explicitly computed, not being deducible from the single particle ones. A way in which progress can be made is by the methods of an apparently far subject, namely Constructive QFT. The reason is that thermodynamical averages at zero temperature, like the conductivity, can be expressed by Euclidean functional integrals of the form

$$\int P(d\psi)e^{V+B}$$

where $P(d\psi)$ is a Gaussian grassmann integration , V is an interaction quartic in the fields and B is a source containing external fields. Such objects (and their bosonic counterpart) are exactly the ones used to represent QFT models. In the case of fermions on the honeycomb lattice the similarity with QFT is even more striking. In Fig. 5 is represented the energy dispersion relation of the Haldane

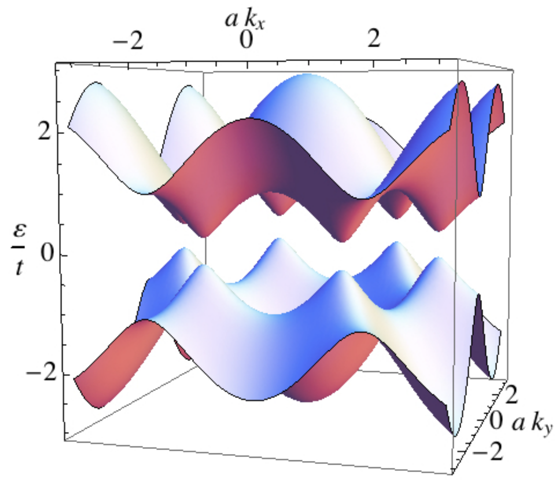

Figure 5: Dispersion relation of the Haldane model

model, and one can notice that it resembles closely the relativistic energy of a massive relativistic particle, that is $\pm\sqrt{v_F^2|\vec{k}|^2+M^2}$, the Fermi velocity $v_F=\frac{3}{2}t$ playing the role of the light velocity. In the case of graphene the gap closes and the dispersion relation is similar to the one of massless Dirac fermions $\pm v_F|\vec{k}|$. As a consequence of this fact, the functional integrals for the interacting graphene or Haldane model resemble regularizations of system of Dirac particles with a current-current interaction. It is also somewhat natural to relate universality in condensed matter to the physics of quantum anomalies in QFT, which display also universality features [19], see e.g. [20], [21].

In the case of models with exponentially decaying covariance, like the Haldane model, convergence of the series expansion for the corresponding funcitional integrals is valid uniformly in the thermodynamic limit; this can be achieved for instance using a a convenient representation in terms of sum of determinants and Gram bounds, see e.g. [22], [23]. In the case of graphene, instead, the covariance has power law decay, and one needs a multscale analysis based on Renormalization Group ideas. One uses the basic property that the sum of grassmann gaussian variables is still gaussian; this allow to perform the integration in several steps, integrating fields of lower and lower energy scale. When the fields with higher energy scale are integrated out, one gets an expression similar to the initial one, with the difference that the covariance is restricted to lower energies and the interaction V is replaced by a different expression called effective potential [28] (see also [29]). In this case, the coupling of the effective potential is not anymore what we called U, but is smaller and smaller at each iteration. In other words, at low momenta the theory is closer and closer to a non-interacting one.

More complicated is the case of the edge states of Hall insulators, which is a model of interacting fermions in one spatial dimension. A typical system of this kind is the Gross-Neveu model with $N = 1$ or massless Thirring model. The $N > 1$ Gross-Neveu is asymptotically free, that is the strength of the interaction decreases at each scale [24]; in contrast, the $N = 1$ case, constructed in [25], [26], [27] is such that the effective coupling remains close to the initial value at each scale as consequence of subtle cancellations in the renormalized expansion, and the same happens in the edge states of topological insulators.

The above considerations show that it is quite natural to face the universality problems in statistical physics using methods inspired by Constructive QFT. First applications of such methods were the proof of the universality of next-to-nearest neighbor 2D Ising model [30] (see also [31]) and of the universal scaling relations in models like the 8 vertex or the Ashkin-Teller model, see [32], [33], [34].

IV. WARD IDENTITIES AND GRAPHENE

Fig. 2 represents experimental data showing that the optical conductivity of graphene at zero temperature is universal and equal to its non-interacting value, that is insensitive to the presence of many body interaction. Why is it so? A Renormalization Group analysis show that the system is closer and closer to a non-interacting theory integrating the high energy modes. This fact is however not sufficient by itself to explain the phenomenon, as interactions can and actually do modify in general the value of physical observables, as it is experimentally seen for the Fermi velocity.

The answer relies on a subtle interplay between regularity properties of the Fourier transform of the Euclidean current correlations and lattice symmetries, and is the content of a theorem proved in [35], [36] and which will be briefly reviewed below.

Theorem *For U small enough the graphene zero temperature longitudinal conductivity computed by Kubo formula with Hamiltonian $H_0 + V$ is*

$$\sigma_{lm} = \frac{e^2}{h}\frac{\pi}{2}\delta_{lm}$$

The starting point is Euclidean representation of Kubo formula. If $\hat{K}_{lm}(p_0, \vec{p})$ is the Fourier transform of the Euclidean current-current correlation $< J_i(\mathbf{x}) J_j(\mathbf{y}) >_\beta$, where $\mathbf{x} = x_0, \vec{x}$, $\langle O \rangle_\beta = \frac{\mathrm{Tr} e^{-\beta H} O}{\mathrm{Tr} e^{-\beta H}}$ and J_i is the current in the i direction, $i = 1, 2$, then

$$\sigma_{lm} = -\frac{2}{3\sqrt{3}} \lim_{p_0 \to 0^+} \frac{1}{p_0}\left[\hat{K}_{lm}(p_0, \vec{0}) - \hat{K}_{lm}(0, \vec{0})\right]$$

In system with an extended Fermi surface the above quantity is infinity and the same happens in one dimension. The fact that the longitudinal conductivity in graphene is finite follows from dimensional considerations relying on the fact that the Fermi surface is pointlike and the spatial dimensions are 2; this implies that $\hat{K}$ is continuous in the momenta.

The correlations in coordinate space can be written as power series in U; convergence follows by multiscale analysis combined with determinant bounds. By such expansion one obtains that the correlations decay for large distances with a power law decay with power 4. From this bound one could not even conclude the finiteness of the conductivity, connected to the derivative of the 3-dimensional Fourier transform. The fact that the integral expressing the Fourier transform is not absolutely convergent is not accidental; indeed the density correlation are even in the coordinates, therefore if its derivative were continuous, then it would have to vanish; no finite conductivity would be present. It is however possible to perform a suitable resummation of the convergent expansion obtaining the following expression

$$\hat{K}_{lm}(\mathbf{p}) = \frac{Z_l Z_m}{Z^2}\langle \hat{j}_{\mathbf{p},l}; \hat{j}_{-\mathbf{p},m}\rangle_{0,v_F} + \hat{R}_{lm}(\mathbf{p})$$

where $\langle \cdot \rangle_{0,v_F}$ is the average associated to a non-interacting systems with Fermi velocity $v_F(U)$ and Z_l, Z are non trivial analytic functions of U; moreover $\hat{R}_{lm}(\mathbf{p})$

is continuously differentiable while the first term is not. In the non interacting case $U=0$ then $R=0$, $Z=1$, $Z_l=v_F(0)$; apparently the conductivity depends on $v_F(0)$ but an explicit computation shows that it is equal to $1/4$ (in units $\hbar=e=1$).

The parameters Z_l, Z, v are expressed by non trivial series in U ; for instance the Fermi velocity is given by $v_F = 3/2t + aU + O(U^2)$ with $a>0$, that is $v_F(U) > v_F(0)$ in agreement with experiments. Universality is present only if intricate cancellations occur between them, impossible to check directly by the series expansion. Note in passing that such a difficulty is rather peculiar; in a QFT model the velocity would be not affected by the interaction, while here it is and despite this fact universality of conductivity must persist.

Universality at the end follows from the following two facts. First of all, the contribution from R to the conductivity is exactly vanishing; it is differentiable and even. The second crucial point is the validity of Ward Identities, that is non trivial relations between correlations following from the conservation of the current; in particular

$$\sum_{\mu=0}^{2}(i)^{\delta_{\mu,0}}p_\mu \hat{G}_\mu(\mathbf{k},\mathbf{p}) = \hat{S}_2(\mathbf{k}+\mathbf{p}) - \hat{S}_2(\mathbf{k}).$$

where $\hat{S}_2$ is the Fourier transform of the average of two Fermi fields, and $\hat{G}_\mu$ the average of two Fermi fields and the density $\mu=0$ or currents $\mu=1,2$. By inserting the explicit form for such averages one gets the following relations

$$Z_0 = Z\ , \qquad Z_1 = Z_2 = v_F Z\ .$$

Therefore the first term reduces to $v_F^2\langle \hat{\jmath}_{\mathbf{p},l}; \hat{\jmath}_{-\mathbf{p},m}\rangle_{0,v_F}$, that is the same expression as in the non interacting case but with a different velocity. But such expression is independent from v_F, and as we said there is no contribution from R; hence universality follows.

Note that the proof makes transparent why in experiments only the conductivity is universal while other observables are not. The theorem above is one of the few rigorous results in the theory of graphene, and helped to settle some debate in the physical literature [17].

V. INTERACTING HALL CONDUCTIVITY

Let us consider now the effect of the interaction on the Hall conductivity σ_{12} of the Haldane-Hubbard model. In the non interacting case we have seen that are there three are different phases in the parameters space, with Hall conductivity which is vanishing or perfectly quantized with value $\pm 1/\pi$. In presence of an interaction, the spectrum of the many body problem cannot be exactly computed. Despite this fact, a rather complete universality picture can be derived, as explained by the following theorem [37].

Theorem *In the Haldane-Hubbard model $H = H_0 + H_1 + V$, for small U there exist functions $m^R_\omega = W + \omega 3\sqrt{3} t_2 \sin\phi + \delta_\omega$, with $\delta_\omega \neq 0$, analytic in U and continuously differentiable in W, ϕ such that*

$$\sigma_{12}(U) = \frac{1}{2\pi}\left[\mathrm{sign}(m^R_+) - \mathrm{sign}(m^R_-)\right]$$

On the critical line, the longitudinal conductivity is $\sigma_{ii} = 1/8$.

A many body interaction destroys the single particle description and it could produce new phases, mainly close to the critical lines where gap vanishes. However, the above theorem excludes the above scenario; the Hall conductivity remains perfectly quantized and universal, the only effect of the interaction being the modification of the critical lines. The result is in agreement with [18], where it has been showed using a topological approach that it if the interaction does not close the spectral gap above the ground state, then the Hall coefficient remains the same as the one of the non interacting case. As it can be seen from Fig. 5, the region where the conductivity is non vanishing is enlarged by a repulsive interaction.

The proof is again based on the combination on Constructive QFT methods and Ward Identities. A power series expansion cannot prove a statement like the above one; as the critical lines moves, necessarily the estimate of the convergence radius tend to vanish going close to the non interacting critical lines. Therefore, a power series approach can lead at best to result far from the lines, well inside the topological regions. In order to get results valid in all the parameter space one writes in the Hamiltonian the masses $m_\pm = m^R_\pm + \delta_\pm$, where $\delta_\pm$ are chosen as functions of $m^R_\pm$ and U so that the results are uniform in m^R; inverting the above relation one gets the renormalized critical lines. This idea is borrowed from the theory of critical phenomena; in Ising models the critical temperature is shifted by the interaction, and one can study the critical behaviour by suitably choosing a counterterm.

The proof of universality starts from the identity

$$\widehat{K}^{\mathrm{R}}_{i,j}(\mathbf{p}) = \widehat{K}^{\mathrm{R},0}_{i,j}(\mathbf{p}) + \int_0^U dU' \frac{d}{dU'} \widehat{K}^{\mathrm{R},U'}_{i,j}(\mathbf{p})$$

where the last term is proportional to the average $\widehat{K}^{\mathrm{R}}_{i,j,V}$ of two currents and an interaction. Some manipulation of the Ward Identities, valid if correlations are at least three time differentiable, implies that

$$\frac{\partial}{\partial p_0}\widehat{K}^{\rm R}_{i,j,V}((p_0,\vec{0}),(-p_0,\vec{0})) = \frac{\partial}{\partial p_0}\left[p_0^2\frac{\partial^2}{\partial p_i \partial q_j}\widehat{K}^{\rm R}_{0,0,V}((p_0,\vec{0}),(-p_0,\vec{0}))\right] .$$

The right side vanishes as $p_0 \to 0$ and this implies that the contribution of such terms to conductivity is vanishing; this implies that all the interaction corrections to the Hall conductivity are vanishing. The regularity property of $\widehat{K}^{\rm R}_{i,j,V}$ in momentum space, on which the cancellation is based, is due to the exponential decay of the correlations in coordinate space, due to the presence of a gap. The

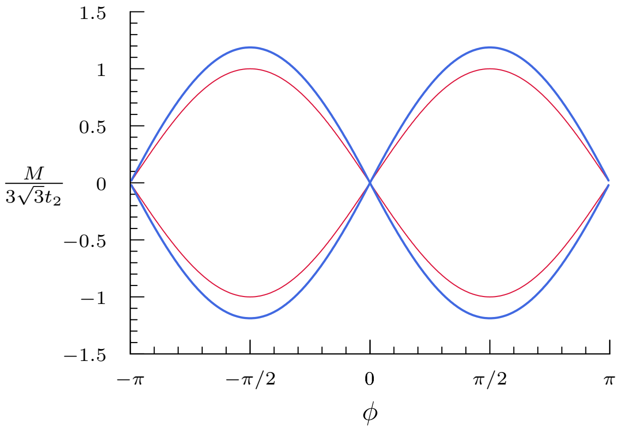

Figure 6: Red: $U = 0$. Blue: $U > 0$.

above use of Ward Identities resembles what is done in QED, see e.g. [20], but here a non-perturbative result is proved.

The proof relates universality in the Hall conductivity to the good regularity properties of the correlations in momentum space, due to the presence of a gap in the (renormalized) non interacting Halmiltonian. This property is of course not necessary for the universality of transport coefficients, and can be present also in gapless systems. We already have seen an example in the longitudinal conductivity in graphene, in which the Euclidean correlations decay with a power law.

In gapless one dimensional systems the interaction has a much more dramatic effect than in Chern insulators or in graphene, qualitatively modifying several physical properties. The conductivity is infinite, the current correlations appearing in Kubo formula being non continuous as function of the frequency (in graphene are continuous and non differentiable). On the other hand, the edge states of topological insulators are one dimensional, and if the bulk-edge correspondence is valid also in presence of interactions, the edge conductance must be universal even in presence of interactions. This actually happens to be true but the mechanism explaining this is much more subtle and is generated by a complicated interplay of lattice Ward Identities and properties of the anomalies of the emerging QFT description, somewhat similar to what happens in Luttinger liquids [34] ,[38].

VI. TOPOLOGICAL INSULATORS AND CHIRAL ANOMALIES

The Kane-Mele model, a paradigmatic system for topological insulators, is obtained summing two Haldane models describing particles with spin $\sigma = \pm$ and with parameters ϕ and $-\phi$. In the non-interacting case the Hall conductivity is vanishing while the spin Hall conductivity is quantized and equal to $\sigma^s_{12} = \pm\frac{1}{\pi}$. With periodic boundary conditions in the 1 direction and Dirichlet boundary conditions in the 2 direction there are eigenfunctions exponentially localized at the boundary. Such edge states can carry a current, whose conductance is the same as the Hall conductivity, a phenomenon known as bulk-edge correspondence.

The edge transport properties can be obtained by suitable limits of $G_{h_1,h_2}(\underline{p}) = \sum_{y_2=0}^{a}\sum_{x_2=0}^{\infty}\langle \hat{h}_{1,\underline{p},x_2}\,; \hat{h}_{2,-\underline{p},y_2}\rangle$, with $\underline{p} = p_0, p_1$ and $\langle ...\rangle$ the limit $\beta\to\infty$ of the thermodynamical average; such expression measures the response of an observable $\sum_{y_2=0}^{a}\hat{h}_{2,-\underline{p},y_2}$ localized in a strip of width a around the boundary $y_2 = 0$ to an external potential proportional to $\sum_{x_2}\hat{h}_{1,\underline{p},x_2}$. Introducing the charge or spin density ρ^i, $i = c, s$ and the charge or spin currents j^i_1, the edge spin conductance is given by $\sigma^s = \lim_{p_0\to 0^+}\lim_{p_1\to 0} G_{\rho^c, j^s_1}(\underline{p})$. Other interesting physical quantities are the susceptibility $\kappa^i = \lim_{p_1\to 0}\lim_{p_0\to 0^+} G_{\rho^i,\rho^i}$ and the Drude weight $D_i = \lim_{p_0\to 0^+}\lim_{p_1\to 0} G_{j^i_1, j^i_1}$.

The interaction modifies dramatically the correlation decay of the edge correlations, with the presence of anomalous exponents η which are non-trivial function of the interaction strength. Despite this strong effect of the interaction, the spin edge conductance is still perfectly universal and quantized; the bulk-edge correspondence holds true also in presence of interaction, as stated by following theorem, proven in [39]

Theorem *For U small, the edge spin conductance in the Kane-Mele model plus an interaction V s*

$$\sigma^s = \pm\frac{1}{\pi}$$

Moreover, the Drude weights and the susceptibilities satisfy the relations:

$$\kappa^c = \frac{K}{\pi v}\ , \quad D^c = \frac{vK}{\pi}, \kappa^s = \frac{1}{K\pi v}\ , \quad D^s = \frac{v}{K\pi}$$

with $K = 1 + O(U) \neq 1$ and $v = v_F + O(U)$. Finally, the 2-point function decays with anomalous exponent η which is related to K defined above by $\eta = (K + K^{-1} - 2)/2$.

The spin edge conductance is still perfectly universal and quantized but the other thermodynamical quantities are non trivial functions of the coupling. They verify however universality relations known as Haldane relations, saying that $\kappa^i/D^i v^2 = 1$ and $\eta = (K + K^{-1} - 2)/2$. Therefore the interacting Kane-Male model exhibits two kinds of universality; the spin conductance or bulk conductivity are exactly independent from the interaction while the Drude weight, the susceptibility or the exponents are interaction dependent but verify non trivial universal relations between them; similar relations are true in vertex models or spin chains [34] or in $d = 1$ Hubbard models with repulsive interactions [38].

The proof is based on a multiscale analysis leading to a representation of the correlations as sum of two terms, one of a QFT model and a more regular part, similar to the one already discussed for graphene but with two crucial differences; first the reference QFT of the dominant part lives in a space with $1 + 1$ instead $2 + 1$ dimensions (as effect of the fact that we are studying the edge states), and second the QFT is interacting theory and not a free one. The decomposition has the following form, if $i = c, s$, $\mu = 0$ is the density and $\mu = 1$ is the current

$$\langle h^i_{\mu,\underline{p},x_2} h^{i'}_{\nu,-\underline{p},y_2}\rangle = Z_\mu(x_2) Z_\nu(y_2) \langle J^i_{\mu,\underline{p}} J^{i'}_{\nu,-\underline{p}}\rangle_H + \hat{R}^{i,i'}_{\mu,\nu}(\underline{p}, x_2, y_2)$$

where $\langle\cdot\rangle_H$ denotes the expectations of a QFT model of interacting chiral fermions in $d = 1 + 1$, known as Helical model, with current-current interaction, velocity v and coupling λ_H, and J^i_μ are its density $\mu = 0$ and current $\mu = 1$. Moreover $\hat{R}_{\mu,\nu}$ is continuous in $\underline{p}$ while the first term is not continuous, and $Z_\mu(x_2)$ is exponentially decaying. The functions $\lambda_H, Z_\mu, \hat{R}$ are expressed by complicate series in U, depending on all microscopic details.

The reason why the above equation, representing the correlations of the interacting Kane-Mele model as the correlations of the Helical model plus corrections, is useful is that the Helical model verifyes extra symmetries with respect the Kane-Mele model; in addition to a global phase symmetry a chiral phase symmetry is valid. There is therefore an extra set of Ward Identities for the Helical model, allowing to get closed expressions from which exact expression for the correlations can be derived. Ward Identities in lattice models are simple consequences of the conservation of the currents. Things are more subtle in a QFT like the Helical model; Ward Identities may not coincide with the ones one would naively guess from conservation law, for the possible presence of extra terms known as anomalies. In particular, if $J^c_0 = J^s_1 = \rho_+ + \rho_-$ and $J^c_1 = J^s_0 = \rho_+ - \rho_-$, $\rho_\pm$ being the fermionic densities, one gets, if $\varepsilon_c = +$, $\varepsilon_s = -$

$$-ip_0 \langle \hat{J}^i_{0,\underline{p}}\,; \hat{\psi}^-_{\underline{k}+\underline{p},\sigma} \hat{\psi}^+_{\underline{k},\sigma}\rangle_H + p_1 v \langle J^i_{1,\underline{p}}\,; \hat{\psi}^-_{\underline{k}+\underline{p},\sigma} \hat{\psi}^+_{\underline{k},\sigma}\rangle_H = \frac{\sigma}{Z(1-\varepsilon_i \tau)} [G_2(\underline{k}) - G_2(\underline{k}+\underline{p})]$$

where $G_2(k)$ is the average of two Fermi fields, $D_\sigma(\underline{p}) = -ip_0 + \sigma v p$ and $\tau = \frac{\lambda_H}{4\pi v}$; similar Ward Identities hold for $< \hat{\rho}_{\underline{p},\sigma}\,; \hat{\rho}_{-\underline{p},\sigma} >_H$. In the above identity τ is

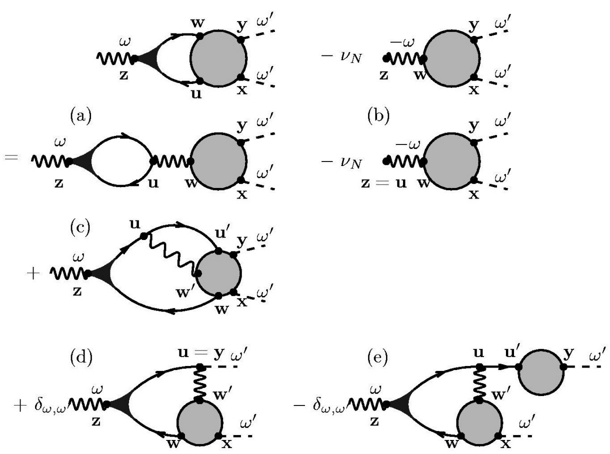

Figure 7: Decomposition of the correction terms to the Ward Identities.

the anomaly, and its origin can be traced to the presence of the regularization necessary to define the functional integrals defining a QFT like the Helical model. Remarkably the anomaly τ is linear in the coupling λ_H; all possible interaction corrections at higher orders are vanishing. This fact is analogous to a well known property of QED in 3+1 dimensions known as Adler-Bardeen non renormalization theorem [19]; for $d = 1+1$ models it has been proved at a non perturbative level in [40]. Some ideas of the proof can be grasped from Fig.7: the corrections to the Ward Identity due to cut-offs can be written as truncated expectations written as sum of terms as in the r.h.s.; the contributions (c),(d),(e) are vanishing in the limit of removed cut-off and only (a) survive.

The vertex functions of the interacting Kane-Mele model are proportional to Helical ones $\langle \hat{J}^i_{\mu,\underline{p}}; \hat{\psi}^-_{\underline{k}+\underline{p},\sigma}\hat{\psi}^+_{\underline{k},\sigma}\rangle_H$ times a multiplicative factor Z^i_μ; by comparing the Ward Identities for the Helical model model with the ones for the Kane-Mele vertex functions one gets the following relations, if $Z^i_\mu = \sum_{x_2} Z^i_\mu(x_2)$

$$\frac{vZ^i_0}{Z^i_1} = 1 \qquad \frac{Z^i_0}{Z(1-\varepsilon^i\tau)} = 1$$

In addition, the current correlations of the Helical model can be exactly computed by Ward identities and one gets, up to terms vanishing as $p \to 0$ or $a \to \infty$

$$G_{\rho^c, j^s_1}(\underline{p}) = -\frac{Z^c_0 Z^s_1}{Z^2(1-\tau^2)}\frac{1}{\pi v}\frac{p_0^2}{p_0^2+v^2p_1^2} \quad G_{j^i_1, j^i_1}(\underline{p}) = -\frac{Z^i_1 Z^i_1}{Z^2(1-\tau^2)}\frac{1}{\pi v}\frac{p_0^2}{p_0^2+v^2p_1^2}$$

By noting that

$$\frac{Z^c_0 Z^s_1}{Z^2(1-\tau^2)v} = 1$$

the universality of the spin conductance follows. In addition $\frac{Z^c_0 Z^\sharp_1}{Z^2(1-\tau^2)v} = K$, with $K = \frac{1+\tau}{1-\tau}$, so that also the other relations follow.

VII. CONCLUSIONS

We have seen examples of properties of matter which are universal, that is essentially independent from microscopic details and common to a large class of systems with different microscopic structure. A mathematical theory is quite well developed in absence of interaction, where a single particle description is valid; much less is known in presence of interaction, where collective effects due to the enormous number of particles involved play a crucial role. Interaction is always present in real systems so one cannot neglect it to explain universality properties. We have reviewed some recent results proving universality in a number of paradigmatic models; the results are based on the rigorous control of functional integral combined with subtle cancellations coming from Ward Identities. Functional integrals appear to be similar to the ones appearing in QFT, and universality follow in this approach via exact conservation laws and properties of the emerging QFT description.

The theoretical understanding of universal phenomena in quantum matter, and its relation with QFT, is surely at its beginning. Main limitations are at the moment the need of weak and short range interactions, while in real materials often strong and long range interactions are present. Fractional Hall effect or the maybe simpler problems related to universal conductivity properties in graphene with Coulomb interactions or in Weyl semimetals are major problems that still need a full theoretical understanding.

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